

# THE MATHEMATICAL GAZETTE

EDITED BY

T. A. A. BROADBENT, M.A.

ROYAL NAVAL COLLEGE, GREENWICH, LONDON, S.E. 10

LONDON

G. BELL AND SONS, LTD., PORTUGAL STREET, KINGSWAY

---

VOL. XXXVII

SEPTEMBER 1953

No. 321

---

## SCHOOL MATHEMATICS TODAY AND TOMORROW

PRESIDENTIAL ADDRESS TO THE MATHEMATICAL  
ASSOCIATION, APRIL 9, 1953.

By K. S. SNELL.

ABOUT a year and a half ago, when the Council did me the very great honour of nominating me as president of this Association, there had just been delivered a series of broadcasts in which eminent men had written letters to be sealed for opening by their successors in 100 years time. It set me wondering what would be taught in schools, say in 50 years time, under the heading of mathematics. The following picture, fanciful in places, came to my mind.

In the lower forms of Junior schools, following the present custom in Nursery schools, "Activity" will be the order of the day, designed by either child or teacher, and there will be no set subjects. In the course of this activity number sense will develop and the four rules of arithmetic will be used—there will be number games, but nothing as exacting as the learning of tables. When subjects emerge, mathematics will appear on the time-table, but no separation into arithmetic, algebra, geometry and trigonometry. All manipulation will be simplified, and in particular the arithmetic. For measurements of all kinds, weights, money will be reckoned in a decimal system, so that gone will be the need for tables of compound quantities, with numbers like  $30\frac{1}{2}$ , the number of square yards in a rod I believe, and gone will be complicated methods of long multiplication of compound quantities.

Even fractions will have given place to decimals, except for a few essential but archaic remnants like  $\frac{1}{2}$  and  $\frac{3}{4}$ . Each mathematical room will have its calculating machine, and the child on duty for the day will do any calculation needed by members of the form. The junior school will thus have time to be concerned with mathematical ideas, functions, graphical representation, with ratio and rate, even with early ideas of differentiation and integration of functions. As geometrical work the child will be concerned with models of solids, and with making them. He will then learn certain facts, such as the theorem of Pythagoras, and trigonometrical ratios, and use these as a basis for calculation and practical work. "Proving" results will have vanished, as completely as has Euclid's sequence today. All geometrical ideas will have become the

tools of trigonometry, which in its turn will very much be dependent on machine calculation.

Turning to specialists in schools, the only geometry will be by analytical methods, so that an early start will be made on abstract geometry. The drawing of figures will be quite out of place, and applications to spatial geometry will not be to two or three dimensions, but to the four-dimensional space-time continuum. Methods of pure geometry, such as inversion and reciprocation, will be regarded as of historic interest, in the same way as the studies of continued fractions and the theory of numbers are in schools to-day. The puzzle interest will be kept alive by the study of "rubber sheet geometry", i.e. topology. Pure mathematics will have become the detailed study of analysis. Following elementary training in functional ideas pupils will study continuity, differentiability, rather than spend time on methods of differentiation and integration of complicated functions. For if these are needed for any practical purposes then machines will produce the required numerical results. I doubt whether even the most elementary trigonometrical formulae will have to be learnt, for even the initial differentiation of  $\sin x$  will be done without the use of sums and products formulae. Algebra, freed from elementary manipulation, will be concerned with the study of groups, sets, and lattices. In applied mathematics, Mechanics will have become an entirely theoretical study, all practical problems having been dismissed at an early stage or passed to the engineer, who will by then be in a separate school faculty. It may be that Newtonian mechanics will still be needed for the initial stage of the pupil's studies, but such will be the demand for results concerned with high velocities and with inter-planetary travel, that I expect a systematic study of Einstein's theory will be necessary at school, if that in its turn has not been replaced by a newer theory. Applied mathematics at school will no longer be confined to mechanics, but will include the study of statistics, hydrostatics, electrostatics, and nuclear fission.

I safeguarded myself at the beginning by suggesting that part of my picture might be fanciful. Let me now consider certain aspects of what I have put forward in a more realistic way, and be concerned with Teaching, as it is today, and may be tomorrow. In doing so I will start at Junior schools and the lower forms of secondary schools. The teaching committee of this association gives a lead here. A report on mathematics in Primary schools has been prepared and is likely to be published in the next year. A preliminary report on teaching in Secondary Modern schools has been published and more detailed suggestions are in preparation. A report on teaching in Secondary Technical schools has been published. Hence I am venturing here, warily, into a domain which is being explored by experts, and I fully realise that my tread may be clumsy. But the importance of this stage is so great that I will take the risk, if only to arouse interest and raise discussion in this direction.

A child's first knowledge of number develops through practical achievements, and through games leading to counting and easy measurement. In the activity periods, which I expect will occupy up to half of the time in school, the child will find the need of basic skills of arithmetic, and hence in the mathematical periods he will need to learn tables and the fundamental rules. Every child gains satisfaction in getting sums right, so that it is an easy subject in which to maintain some interest. This satisfaction can be nurtured by keeping the sums sufficiently easy, even while progressing to new types. It is a fact that teachers demand to an amazing extent many easy examples, without their being necessarily original or interesting, simply because children like going ahead with examples which they can do, and may get right. Or is it that teachers are too content with this way of keeping them quiet? In addition to this arithmetical work, there is the recognition of shapes which

leads a child to the making of models, and knowledge of the basic geometrical figures. This needs simple measurement, and leads also to an informal introduction to mensuration and early geometrical facts.

It is when we come to fractions that we soon step outside the child's natural orbit. Why must he learn to add, let alone multiply, fractions like  $\frac{5}{6}$  and  $\frac{2}{3}$ , and much harder fractions, when the corresponding process in decimals is so much easier? It is also so much more logical, following the system of numerals based on the position of digits in a number, to work in decimals. The work in fractions is due in part to our system of units, which do not adapt themselves to decimals, and so must use fractions. I would like here to consider seriously the possibility in this country of a greater, and eventually a complete, use of the metric system. The various units of money, length, weight, capacity have developed along sensible lines, and it is of historic interest to trace the origin of units like a furlong, an acre, a quart. But they do involve complicated calculations, which would be unnecessary if we used a decimal system. Think firstly what extra work is involved to the child. The 12 times table is needed for conversions of pence to shillings, and inches to feet, and if such units did not exist children would probably not learn the 11 and 12 times table as a normal routine. They have to learn to add compound quantities in  $\text{£ s d}$ , in yd., ft., in., in lb., oz., all different and needing additional knowledge. They learn to multiply and divide these compound quantities, at any rate by simple numbers, and long multiplication and division both involve complications in setting out intelligently, and so have been postponed, or omitted, from the necessary work. The fact that there is no simple relation giving the weight of water, as in the metric system, means additional work, and causes more difficulty in understanding the meaning of density and specific gravity. Moreover in all mensuration work there have to be examples in both the English and metric system, again involving extra time, and obscuring principles behind complications in working. The Secondary Modern School report reminds us that "compound tables are not of basic importance as *mathematics*"; teaching them is justifiable only in proportion to their everyday usefulness". Let us then consider what possibility there is of conversion to a metric system. In earlier times most countries had their own units, but this was confusing, and hence in the nineteenth century the countries of Europe adopted the metric system. The English system was already widespread since it was used throughout the empire. That meant that the difficulty of conversion was considerable and would involve changes over so wide an area. But the metric system may be said to be infiltrating into use. All scientific work uses the metric system. It is used for lengths in our ordnance survey maps, and throughout the Services. Even in measuring angles the degree is divided into decimal parts, both in the American and sometimes the British artillery. What changes exactly would be involved? For length we are already becoming familiar with the metre, which is near enough to our yard to be a useful unit—international sports meetings have familiarised our people with lengths like 1500 metres, and 5 and 10 km. This would also necessitate a change to the metric system for areas and volumes, which would be more difficult. In the English countryside we are familiar with acres, and farmers are notoriously conservative. Again the quaint 4, 2, 4 table for gills, pints, quarts, gallons is firmly ingrained in our social life, though litres are becoming increasingly familiar through references to capacity of all internal combustion engines. For money the conversion would be simple in theory, though it would involve some change in our coinage, and hence would encounter considerable resistance, and need propaganda to convince people of the value of the change. Taking the  $\text{£}$  as unit, a florin is a "deci-pound", and a new coin like our threepenny bit, but in value  $1/100$  of a  $\text{£}$ , about  $2\frac{1}{2}\text{d.}$ ,

would be the "centi-pound", leaving the farthing as the "mille-pound". With the present variability in the value of our money this is surely an appropriate time to suggest a change such as I have outlined. It might then be possible for buses to save odd fares like  $2\frac{1}{2}$ d.,  $3\frac{1}{2}$ d. Is then such a change possible, and is this the time to press for something to be done? There are many interests involved and I have considered it only from an educational outlook. If we in this Association think that it is important we could be one of many bodies pressing for an advance.

I shall now consider various sections of the mathematical course in schools and make some suggestions on them. First I turn to Geometry. However much we unify our course there are pieces of knowledge, and techniques, in geometry which are different from those in any other section of the subject. Further there has been so much change in its presentation and subject matter over the last fifty years that we shall do well to consider if we are travelling in the best direction, or whether we are allowing the momentum gathered in the change to take charge and carry us too far in one way. Briefly I am going to suggest that we are now laying overmuch emphasis on the practical to the exclusion of the more theoretical deductive work, and at the specialist stage are putting overmuch emphasis on analytical work to the exclusion of the methods of pure geometry.

Let us start then in the Junior schools. There used to be no geometry in the old "Elementary" schools. Now the recognition of spatial properties is seen to be an important part of the education of the young. Fitting together different parts of an object, and placing shapes in correct spaces are used as toys from early days. Modelling with plastic material is at first imaginative, but becomes more exact as a child wants special models, which have to be the right size, and thus easy measurement is needed, and reducing scales. Starting thus from models of "real" objects a child soon gets familiar with standard geometrical objects, cubes, pyramids, cones, and wants to make them. Modelling with plastic material changes to making solids out of paper or cardboard, and this brings the need for more accurate measuring and drawing, and hence for the use of instruments. Thus the child as well as learning names of standard solids will also become familiar with plane figures, rectangles, triangles and so on. This work is essentially practical to the child, but the teacher will note the growth of geometrical knowledge, and will be ready to stress results as they appear. With sufficient models available it must be interesting for a group to tabulate the numbers of faces, edges, and vertices of different solids, and to see if they discover the general relation connecting them. Even at this early stage, while the child has free play, the teacher is guiding, and helping to systematise the knowledge the child is gaining. Our education tends now to provide more and more for the ordinary child, and he may not be attracted to general results, but the teacher must also remember that there are the few who are going to be mathematicians, and who may receive inspiration through this early work. It is surprising how much a child can do with guidance, and I was very much impressed this year by a display of mathematical models made and skilfully displayed by boys of 13 and under in a preparatory school.

In secondary schools, as part of the general mathematical course, the geometrical emphasis is on drawing and calculation. The early introduction of trigonometrical ratios, which is rightly advocated in the trigonometry report, and also the early knowledge of the result of Pythagoras' theorem, has opened to teachers a wealth of useful practical examples. These appeal to boys and girls and are in general well within their capabilities. Also this kind of work fits in to a mathematical course, and serves to unify the mathematics, an object rightly aimed at. The teacher also finds such work easier to present

than any form of rider work in geometry. Hence there is a tendency for calculations to occupy a very large part of the geometrical course. But does this give the best mathematical training, and is this giving the boy the mathematical ideas which he needs? Rather does it not lead a boy to regard calculations as the main aim, and any theoretical deductive work as a means to this end? I suggest this is a phase through which boys must pass, and we in teaching must place temporary objectives before them. Scale drawing can be succeeded by calculation of lengths and angles, and that again can and should be superseded by deductive work of a more theoretical nature or by a generalisation from particular results. I know that I am thinking in terms of pupils in grammar schools in particular, because I am looking always for potential mathematicians, rather than technicians, but my thoughts are shared by the writers of the Secondary Modern Schools Report. Listen to the following quotation:

"Some children who are not primarily regarded as mathematically gifted respond surprisingly to the appeal of mathematical truths of a comparatively advanced nature. This is particularly true of Geometry, where visual intuition may enable a child to grasp and appreciate a fact whose logical derivation would be far beyond his ability. For example the graphical examination of Pascal's 'Mystic Hexagon' has been found quite interesting to Modern School boys. Apart from the curiosity engendered by unsuspected and, to the child, unlikely properties, there appears to be also a sense of satisfaction that these properties consistently 'come out right'; it is suggested that the child can thus come to appreciate in some degree the beauty of the generalisations which emerge as underlying laws." This suggests the value of looking for general theoretical truths at an early stage. Again Secondary Technical schools will naturally look to Geometry for its practical aspects, but in the report for these schools issued by the Association I find the following: "Although many of the exercises will be numerical, the rider robbed of its formal cloak and reduced to simplified statements of steps will have its place in the course." It will be argued, correctly, that calculations in Geometry demand deduction, and this is especially true of some work in solid geometry, in drawing nets of solids for instance, or in drawing simple plans and elevations, the value of which is emphasised in the report from which I have just quoted. But in both of these reports, which are concerned with schools where the intellect of the children may not be as high as in grammar schools, the value of generalisation and of theoretical deductive work is stressed.

There is another influence which is affecting the teaching of Geometry. The recommendation that Ordinary level Certificate papers should be on general mathematics rather than on individual subjects, has enabled some teachers to omit all work on riders as being a "bad bet" for those they are teaching. Again the "alternative syllabus" proposed by the "Jeffery report", and now largely adopted, introduces new topics, and something has to be curtailed to make way for these; and this, unintentionally, has left little time for work on riders.

For these various reasons I consider we have been turning in a wrong direction, certainly in secondary grammar schools, in our teaching of Geometry. I believe that rider work is vital and opens the door of mathematical thought and methods to many. Most teachers in schools must recall some boys or girls who are weak in most of their mathematics but find deductive work in Geometry more interesting and more within their ability. These are the exceptions, I admit, and rider work is more difficult to cope with in a class than other types of mathematics. Boys and girls, the former especially, tend to find riders tedious to write out, and it is painfully easy for them to produce nothing at all, and teachers certainly find a full Geometry exercise of riders

extremely tedious, especially if they are conscientious in their correcting. Hence let me be practical for a minute and tell you my normal procedure. I never set more than two riders in any written exercise done in school or in preparation. I encourage quick rider work in school, in which boys are required to sketch a figure, mark as much as they can in their figure, write as little as possible, so that at the end of twenty minutes or half an hour when going through the work together from figures which I have drawn on the board, any boy can tell me how he solved the question. If the riders are easy enough they may easily do five or six in a lesson in this way. Again interest is more easily aroused when the result proved is striking and unexpected, and questions can sometimes be set in the form—what can you find out about lengths, or angles, in this figure? A particularly interesting result, such as Simpson's line, or a locus like Apollonius' circle, can be found experimentally, or if necessary quoted, and a series of one-step riders given to lead up to that result. I apologise for suggestions which are probably familiar practice with most schoolmasters. But members of this association have done much in the past to abolish rigid adherence to Euclid's sequence, and we, in our turn, must ensure that this does not detract from the quantity of deductive work done in schools. Rather, with the added freedom that we have gained, we should realise the inspirational value and importance of all such work in Geometry. I will close this section with a quotation from Mahatma Gandhi, taken from his autobiography: "When, however, with much effort I reached the thirteenth proposition of Euclid, the utter simplicity of the subject was suddenly revealed to me. A subject which only required a pure and simple use of one's reasoning powers could not be difficult. Ever since that time geometry has been both easy and interesting to me."

In passing to the geometry for specialists in schools I must first refer to the third report on the teaching of geometry, prepared by a sub-committee of our teaching committee, and about to be issued to members. This is concerned mainly with geometry taught to the sixth forms, and the unanimity with which it was received by members of the teaching committee suggests that it is to be a worthy successor to the two previous reports, and may serve as an inspiration to all teachers in schools, whether or not they are teaching that stage of geometry. It is therefore presumptuous for me again to tread on ground which has been explored so thoroughly by a team of experts.

However I am going to blunder into it. The changes in geometry teaching at this stage are even greater than in the earlier stage, and I wish to review the position, and make certain suggestions, if only in the hope that it may encourage you to give to this new report the attention it undoubtedly deserves. Thirty years ago the specialist had two subjects, modern pure geometry, and analytical geometry, and they were treated quite separately. In pure geometry he learned further triangle properties, overlapping much trigonometrical work, pole and polar properties for the circle, harmonic ranges and cross-ratio, inversion, reciprocation, and especially point-reciprocation, some solid geometry particularly concerning the tetrahedron, and finished with orthogonal, conical and general projection. This was a most inspiring course, but it did not lead anywhere, except in giving various approaches to the consideration of conics, and I well remember at Cambridge the feeling that there was nothing more to be done along those lines, and that the course lacked any final objective. In analytical geometry there was a new subject, with a course leading through the straight line, the circle, conics referred to their axes, and the general equation of the second degree, and this clearly was a vital subject leading to a university course in three or more dimensions.

In the new course now being worked out these two subjects are correctly blended into one, thus avoiding much repetition, enabling the pure geometry

meth  
and le  
analy  
unive  
carc  
In  
of the  
work  
speci  
direc  
to se  
to cu  
is stu  
and  
that  
are n  
teach  
ahead  
Again  
Cam  
year  
or re  
a two  
boys  
geom  
anal  
to pi  
mech  
from  
geom  
in sr  
it.  
secti  
first  
clear  
and  
can  
proj  
of d  
harr  
Men  
know  
ordi  
som  
more  
O  
can  
tion  
stag  
that  
beli  
whe  
defi  
proj  
inve

methods to be used when required in any topic as it is considered analytically, and leading through spatial geometry to an abstract geometry on an essentially analytical foundation, this last being the geometry now mostly studied at the universities. Such a course is set out in detail in the report and merits most careful study.

In any transitional stage there are dangers of swinging to extremes. One of the dangers I foresee at present is to magnify the importance of analytical work, and belittle the importance of pure geometry methods. A young specialist normally takes to analytical geometry with delight. It gives him direct methods of proving results where before, in pure geometry, he has had to search for ideas as to how to start, and it at once broadens his knowledge to curves other than the circle. It makes much use of the calculus which he is studying at the same time, thus helping him to see the unity of his subject and the interaction of one part on another. It is thus easy to miss the fact that he is at times using very cumbersome methods to obtain results which are much easier to establish by pure methods. Further the human element in teaching arises again here. It is easier for the young boy or girl to work ahead, with occasional guidance, in analytical work, than on pure geometry. Again examination syllabuses have had perhaps an unfortunate effect. The Cambridge joint advisory committee for mathematics issued in 1945 a two year syllabus for mathematics and this did not include any conical projection or reciprocation. This was probably correct since the syllabus was only for a two year course, and made no reference to the third year work which many boys would do at school. This has probably reduced the amount of pure geometry, since the new analytical work occupies so much of the time. This analytical work is appreciated by the student, but the methods he uses tend to produce long and complicated algebra, and to become too stereotyped and mechanical, so that after a time he may lose the inspiration which he can get from pure geometry methods. What does this lead to? I think that pure geometry needs constantly to be kept in mind, and can be done frequently in small instalments, or in definite sections involving longer periods spent on it. I do not think that the young specialist can learn enough by fitting in odd sections of pure geometry in the middle of his analytical work, but that at first he will continue to think of it as a separate though allied subject. A clear starting point for this kind of work is the theorems of Ceva and Menelaus, and through this the student leads to harmonic ranges, and cross ratios. He can then learn the methods of inversion, and of conical, as well as orthogonal, projection. We like to avoid repetition, but it is a good thing to see the use of different methods for proving an important result. As an example the harmonic property of the quadrangle may be obtained by the use of Ceva and Menelaus, by projection into a parallelogram—out of fashion at present I know, but interesting for the student to see—analytically using oblique co-ordinates, or homogeneous co-ordinates, or by cross ratios, and it may be some time before the student appreciates that the last method is better, or more general than that of projection into a parallelogram.

One question a teacher has to face is how soon the more intelligent pupil can pass to purely projective methods, rather than through metrical definitions of cross-ratios. One school of thought suggests that if left to a university stage the student will be ready to start with a purely projective basis, and hence that he should not tackle the subject at school. I would oppose this, as I believe a better appreciation of projective geometry can be obtained gradually when a pupil has been prepared by seeing at some stage that the metrical definitions he learnt are no longer necessary, and that certain results are projective, while others are metrical. He will then be the more ready to investigate a purely projective geometry. I would compare this with the

definition of trigonometrical functions in terms of exponential functions of a complex number. At school a boy can work up to this definition, and realise that he could start afresh and obtain all the familiar properties of the functions from such a definition. The boy is continually obtaining properties, which open his vision to more general definitions, and we must beware of hurrying this process too much. A boy of today is not necessarily more mature than his predecessor of a generation ago, and we cannot expect him to adopt mature ideas before he is prepared. This preparation does then in my opinion include a careful introduction to pure and projective geometry through metrical properties. I would then advocate serious study of the place of pure geometry in the specialist stage at school, and warn teachers to ensure that this most inspiring section of work is not crowded out and so lost.

The next group of subjects I would like to consider are those leading to formal analysis, and they start separately as algebra, trigonometry, calculus, once calculus has emerged out of the general treatment of functions in the mathematical course of the main school. In algebra the move to reduce manipulation has gathered momentum, so that in the main school much that was formerly drilled into the shape of a skill is gone through quickly once the technique has been obtained. In consequence the young specialist at the beginning of calculus often does inaccurate work through faulty manipulations of fractions. The solution of this difficulty is not a reversion to more manipulation in the main course, but the greater need for quick revision and consolidation by the more advanced student, who will learn the more quickly when he sees the need for the work.

Trigonometry has its essential spade work, involving manipulation of the type that the young specialist enjoys. The first course extends to the solution of simple trigonometrical equations, by graphical and other methods. Triangle formulae involving radii of inscribed and circumscribed circles are not an essential part of the main course. They provide excellent work for the enthusiastic scholar who wants to research on triangles, and prepare and investigate a section of work for himself. Similarly in algebra continued fractions and the theory of numbers are omitted completely by the majority, but give excellent scope for an enthusiast wanting to do some work on his own. With the introduction of complex numbers school algebra and trigonometry merge, and in subjects like factorisation, or symmetric roots of an equation, the same principles are involved, whether the function concerned is  $\cos n\theta$  or  $x^n - a^n$ . A schoolboy will rarely advance far into the study of functions of a complex variable, but side by side with the treatment of hyperbolic functions he will be interested in the corresponding treatment of trigonometrical functions, and also of the logarithm of a complex number. An early introduction to more advanced work is always valuable, giving the pupil a vision of what he must do as he advances.

In calculus there are probably three distinct stages in schools. An early introduction to the ideas is rapidly becoming a part of the normal mathematical course of a large section of the grammar school population. The calculus report, issued and discussed over a year ago, shows how this can be done, with the differentiation and integration of simple powers of  $x$ , and a wide field of applications. It is a pleasant climax to a main school course to be able to obtain a result such as the formula for the volume of a sphere, an application which is within this stage.

Once the possibilities of the subject have been realised by the scientist, technical student, or mathematician, he will then be ready for the second stage. This includes the differentiation of other algebraic and trigonometrical functions, and simple methods of integration, and the idea of a definite integral as the limit of a sum. The relation between this and the indefinite

integral obtained by anti-differentiation cannot of course be proved with any rigour, but the limit of a sum idea is very essential in applications such as finding centres of gravity and moments of inertia, and the student should be encouraged to use it freely. This work has the advantage that scientists and mathematicians both need it and so can sometimes work together. Provided assumptions are stated clearly I consider it important that the future mathematician should go ahead rapidly gaining as much manipulative power as he can, and not being held back unduly by considerations of rigour which he will comprehend much more readily as he grows more mature. Calculus at this second stage is the core round which the pure mathematics can be grouped, the necessary algebra and trigonometry being done as required. It is in my opinion the easiest of the three main courses for the young specialist, i.e. easier than the mechanics or geometry. The better scientist can extend this stage to include the interpretation of double and triple integrals, and their evaluation in simple cases. It will also have become apparent to the scientist that a major need for him is the ability to find a function to represent a curve obtained experimentally. Hence he will appreciate an introduction to the formation of Fourier's series.

The third stage of calculus at school is for the more able mathematical specialist. It looks for more rigour, and will eventually give him an insight into ideas of analysis such as the definition of a limit, continuity, and differentiability. But it is no use hurrying this stage for the ideas involved need a mind more mature than that possessed by the normal boy or girl at school. They should advance further in the theory of integration as summation, and the properties of definite integrals afford much opportunity for skill and careful thought. This is the time when they should be able fully to understand how to develop the theory of logarithms from the definition as an integral—often this is attempted much earlier when geometrical methods have to replace knowledge of properties of definite integrals. As an example of the different treatment in stages two and three consider the methods of rectification of a curve. The young boy has no difficulty in the idea of a length of a curve—he has measured the length of a straight line—he learned early the length of the circumference of a circle—and he would think of the length of any curve as that of a piece of cotton that could be wrapped along it. Further, if he has been taught to see the applications of integration by summation he will be ready to approach the problem by thinking of  $\delta s$ , and then of  $\Sigma \delta s$ , and hence  $\int ds$ . To evaluate this he will be accustomed to change of variable by substitution, and so will convert naturally to  $\int \frac{ds}{dt} dt$  if the curve is given parametrically, or in terms of  $x$  or  $y$  otherwise. He will be ready to assume that the ratio (chord :  $\delta s$ ) approaches 1, as  $\delta s$  approaches 0, and since the square on the chord is  $\delta x^2 + \delta y^2$ , he will obtain the normal integral in the form

$$\int \sqrt{\left\{ \left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2 \right\}} dt$$

between appropriate limits. This will enable him to calculate the lengths of curves with known cartesian equations, and is a treatment that can be given before the boy is ready for the formal treatment. It is essential to emphasise the assumptions made, and if this is done, some will be prepared for the further work. But, as the Calculus report says, it is impossible to start a logical theory from  $ds^2 = dx^2 + dy^2$ . Hence a more careful treatment is needed in the third stage, giving the definition of length as an integral, and hence leading to a course of differential geometry, extended to include curvature and evolutes. I think it would be quite wrong to start an

immature student on this treatment, which is described in chapter 10 of the calculus report, or to postpone any consideration of lengths until the student is mature enough for this treatment. Our aim at school in the analysis course should be first to give a wide power of manipulation, and gradually to present more rigorous mathematical ideas as the pupil becomes sufficiently mature to understand them. Some schoolmasters and university lecturers emphasise that a boy must not learn anything that he must unlearn, and hence go on to say that the boy should be introduced to topics through the logical way. If this were carried to its logical conclusion we should go back to the days when a geometry course started with statements of axioms, and many an ardent seeker after mathematical knowledge would be held up, and so have his ardour diminished, by sections of work which he was unable to understand, and had to accept under tuition. I am of the opposite school of thought, and want all the time to go ahead with ideas, stating assumptions, and pointing out the need subsequently for piecing together what has been done in a logical system.

When I turn to applied mathematics, I am at first concerned about the correct distribution of time in the various sections of the subject. I dislike again referring to examinations, but this is a topic in which I consider that examinations are exerting a wrong influence on teaching, and I am anxious to see the way out of the difficulty. Some university examination boards divide the subject mathematics into pure and applied mathematics, two subjects, carrying equal importance, and hence the same number of papers, and it is this last point which makes for difficulties. The reason for this is clear, as there is the reasonable desire to enable these advanced level papers to serve as intermediate examinations for a university course, and thus save the student from taking too many different examinations. But unfortunately, from the point of view of teaching, the subject does not divide up in this way. Rather are there three sections, Analysis, Geometry, Mechanics, and these three demand approximately equal attention for the mathematician, though the scientist and technician need not go as far in Geometry, and hence after a time can spend less time on that section. At a university stage applied mathematics is a far larger subject than at school, including Electricity and Magnetism, and Hydromechanics, and then it is possible that half the examination might be devoted to it. But at school applied mathematics means Mechanics, with possibly some Hydrostatics, and lately some Statistics. A consequence of this examination system is that in setting the appropriate number of questions on Mechanics to fill two, or even three papers, these questions become very artificial, and tend to have a deadening effect on teaching, since schoolmasters are only human in using questions set by their seniors. It is interesting to note that in the final school at Oxford there used to be three applied mathematics papers, two of which were on Mechanics, but that one of these has been dropped, because of the tendency to produce artificial questions to complete the two papers. It must be remembered that at school most boys take a course in experimental physics which includes the foundations for work to be part of their applied mathematics at the university. Hence it is fair that half their time should be devoted to applied mathematics and physics, leaving the other half for analysis and geometry. Another point to be borne in mind is that scholarship examinations for Oxford and Cambridge do not rate applied mathematics as anything approaching one-half of the examination. The problem is difficult, but I think we may say that schoolmasters will continue to divide their time in the way they think best for teaching, and we hope examinations may adjust to fit in more with the practice in schools.

I have postponed till the end my views on the teaching of mechanics because

I have addressed various meetings of the Association, or branches, on this subject, and I do not want to repeat myself more than I can help. As in calculus I can see three stages in the teaching, and since mechanics is the other part of the mathematics course which must be taken by all who have a scientific bent, the three sections follow similar lines. The first stage can be fitted in to the elementary mathematics course in the main school. It can be used to stress the value of certain ideas and techniques which are there developed, and to show their application to practical problems. As a start in statics, the idea of inverse ratio is needed in considering moments, and the geometry and trigonometry of the triangle is used exhaustively in considering forces acting on a particle and the triangle of velocities. Easy kinematics provides one of the best illustrations of the use of graphical methods, and the approach through graphs shows the boy how to deal with questions involving variable velocity and acceleration, rather than stressing too much at first the particular case of constant acceleration. An introduction to dynamics in a straight line naturally follows and the concepts of momentum and kinetic energy, which some boys take a long time to understand fully, have their first introduction at an age when boys are interested in finding out how long, and in what distance, a car can pull up from a large speed, and kindred problems. Such a course is based more on intuition, and a boy's experience, than on experiments, though simple experiments can be introduced as necessary. There are two aspects of force which are used in this course. The first concerns the force needed to hold an object in equilibrium, and such forces are naturally compared to weights and measured in the same units. The other idea of force is that which causes acceleration, and while the first demands to my mind measurement in lb. weight, the latter involves the use of  $g$ , and the fact of  $g$  being variable suggests the need for an absolute system of units, and thus prepares the boy for the use of such a system in his main course. Much of the course I have described here is part of the physics, rather than mathematics, in many schools, but it seems to me much more suitable in the latter.

In the second stage Mechanics emerges as a vital subject at the beginning of specialist work for all mathematicians and scientists, and principles can be developed partly by working from intuition, and partly from experiment. The principles of statics, including the general reduction of forces, and the equilibrium both of a particle and of a rigid body acted on by forces in two dimensions, all seem easy to explain, and in fact one of the difficulties in teaching statics is that there is so little to learn in the way of principles, and once these have been taught one wonders why boys ever find they cannot solve all problems. Problems must be graded carefully because one of the difficulties boys experience is that of drawing a clear diagram and then of choosing the best method to tackle the problem. It is as well to deal thoroughly with problems involving one rigid body, before introducing problems with two or more bodies and the interaction of one on the other. Similarly in dynamics a thorough treatment of the single particle is valuable before dealing with problems of impact. At this stage boys should be introduced as much as possible to problems involving variable acceleration, which can be treated graphically and with the help of calculus. It is surprising to me how long a boy takes to accustom himself to the advance in measuring work done by a force from  $P \times s$  to  $\int P ds$  or the area under a graph, and the longer he deals exclusively with constant forces the greater the difficulty. All ideas in Mechanics seem to sink in slowly, and constant repetition is necessary. If the same ideas can be repeated in new guise then there is an obvious advantage in maintaining interest, and this occurs with the consideration of the motion of a rigid body about a fixed axis, an invaluable new topic for revising all dynamical ideas and for linking up with the appropriate section of calculus.

Problems of increasing complexity encourage too many to continue on these lines, and to limit themselves to discussion of particular problems of motion on the earth's surface, and be content not to examine carefully enough the foundations of the subject. The mathematician must advance to this, the third stage, in school mechanics. The dividing line is not clear cut, and the third can overlap the second stage, while competent mathematicians can start earlier to think of the more theoretical and fundamental assumptions. In these times they will have heard of Einstein's theory of gravitation, and probably have recently read of his unified field theory, and if they are to understand at any time what Einstein has done they must understand the basis of Newtonian mechanics, and what assumptions Newton made which are not accepted by Einstein. They will wonder why, if the tenets of Newtonian mechanics have now been replaced by a more modern theory, they are still learning what is out of date. Such thoughts may drive them back to more careful thoughts about initial concepts, definitions of time which is absolute in Newton's hypothesis, and of distance which is measured relative to any origin, and with no idea of a fixed origin. Popular expositions of Einstein's special theory will introduce them to the idea of the relativity of simultaneity, which is thus contrary to the Newtonian hypothesis of absolute time. The boy must next be encouraged to think clearly about the concept of force, and in particular, weight. Forces caused by the action of one body on another, with which it is in contact, seem definite enough, but how about forces due to attraction through space, such as weight, or the force which causes the earth to travel round the sun? Reference to Newton's law says that weight can be measured as  $mg$ , where  $m$  is a constant of the body, and it must be made clear that this is an assumption made by Newton to fit the facts as observed by Kepler. This was the law which Newton framed to explain what he observed;  $mg$  is the appropriate gravitational force for a frame of reference fixed to the earth, i.e.  $m$  times the observed acceleration of a freely falling body. This idea can be helped by asking what is the appropriate gravitational force for other frames of reference, such as a freely falling lift, or a railway carriage rounding a curve, or for a frame of reference attached to the moon. It can be explained that this assumption led to a theory which fitted the facts with remarkable accuracy, but not with sufficient accuracy when dealing with velocities approaching that of light, and that Einstein's theory gives a different explanation to replace Newton's second law. It is harder to get the young boy to realise that the assumption that space is Euclidean was another which Newton made, and which is now replaced, but it is valuable if he can be brought to see here the vital influence on the theory of the physical universe which has been played by the development of new geometries. I think a boy should leave school introduced to these problems, and wondering about them, but we must leave it to a more advanced stage of his mathematical career to develop the new theories. Our duty in schools is to send students to universities full of desire to investigate further, rather than worn out or deadened with continued repetition of artificial problems.

I have left to the end my attempt to express my gratitude for the honour done to me in appointing me president of this Association for the year. When I look back at the names of the famous mathematicians who have held this position I am the more humbled and grateful for being allowed to stand momentarily in their company. As a schoolmaster I have not given my time to mathematical research, and hence could not address you on any important mathematical topic. I have however maintained a very great interest in teaching, and though what I have had to say has been very plebeian, and necessarily cursory in covering a wide ground, I hope it will prove of some value to other teachers, and will give some information to those of you who are

in universities, as to how we are thinking in schools. It is a very great merit of this Association that it numbers among its members men and women from universities, and from every kind of school, and that at our annual meetings we mix and hear views from the other side. The fact that I have been allowed to preside I take as a very great honour, not only to myself, but to all those of us whose duties lie in the schools.

K. S. S.

## CORRESPONDENCE.

### EUCLIDEAN GEOMETRY AND THE RIGID MOTION GROUP

To the Editor of the *Mathematical Gazette*.

SIR,—In my article on "Euclidean geometry and the rigid motion group" I inadvertently (and irrelevantly to my thesis) identified the rigid motion group with the group of projective collineations which leave invariant a fixed involution on the line at infinity. The rigid motion group is of course a self-conjugate subgroup of this group of collineations.

I am indebted to Dr. D. B. Scott (and other friends) for their kindness in drawing my attention to this lapse. I have no wish to pioneer the discovery that all similar triangles are congruent.

Yours, etc.,

R. L. GOODSTEIN.

SIR,—The use of the word "motion" in both a physical and a metaphorical sense, which Professor Goodstein criticises (*Gazette*, No. 320) is especially dangerous at the stage at which a pupil passes from the study of spatial geometry (physics) to abstract geometry.

Professor D. E. Littlewood writes (*Gazette*, No. 310): "The essential point that all philosophers appear to have missed is that the denial of the principle of superposition does not merely invalidate one theorem, but destroys the whole structure of Euclidean geometry."

The natural interpretation of these words is to suppose that a formal abstract Euclidean geometry is invalidated if it is illegitimate to take up one triangle and to fit it on to another. It is undeniable that it is illegitimate to do so; but anyone who wishes to know why this does not invalidate the theory of an abstract Euclidean geometry will find a complete answer in Forder's *Foundations of Euclidean Geometry*.

Yours, etc.,

CLEMENT V. DURELL.

### GLEANINGS FAR AND NEAR

1746. Particular numerical examples should be used. . . In this kind of way the pupil's mind may be made familiar with the transformation from  $5 = 2 \sin x$  to  $\sin x = 2/5$ . The trigonometry may in fact be used to help teach the algebra.—*The teaching of trigonometry*: a Report prepared for the Mathematical Association (1950), p. 9. [Per Mr. J. T. Combridge.]

# A SYMBOLISM FOR THE GEOMETRY OF THE TRIANGLE.

By R. H. COBB.

THE purpose of this article is to outline a system of notation for dealing with some of the properties of the general triangle. By means of it results obtained may be concisely stated and, by symbolic manipulation, other results implied but not explicitly realised, may be found.

**1. Isology.** Consider the method commonly used for proving a property of the triangle, for example, that the medians are concurrent. We have a figure  $F$  containing a triangle  $\Delta$  with vertices  $A, B, C$  and midpoints of sides  $P, Q, R$ , and by applying the appropriate reasoning to the figure obtain the result that  $AP, BQ, CR$  are concurrent at, say,  $G$ . The validity of the reasoning does not depend upon what figure we have taken, nor upon the names, that is, the letters, used for the various parts of the figure. Thus, if in another figure  $F_1$  there is another triangle  $\Delta_1$  with vertices  $A_1, B_1, C_1$ , and midpoints of sides  $P_1, Q_1, R_1$ , we can assert, without repetition of the proof, that  $A_1P_1, B_1Q_1, C_1R_1$  concur at a point  $G_1$ .

The relation between the two figures—that our reasoning applies equally well to either—is an “isology”; the figures are “isologous”, or one is the “isologue” of the other. The words can also be used with reference to corresponding parts of the figures; thus  $G_1$  in  $F_1$  is the isologue of  $G$  in  $F$ .

**2. Specific parts.** Consider now another property of the triangle, that if a line parallel to  $BC$  cuts  $AB$  and  $AC$  at  $X$  and  $Y$  respectively, then

$$AX/AB = AY/AC.$$

If we have proved this for  $\Delta$  we can, of course, assert the isologous property of  $\Delta_1$ . The difference between this case and the previous one is that  $A, B, C, P, Q, R$  and  $G$  have each a unique isologue in the figure  $F_1$ , but  $X_1$  and  $Y_1$ , the isologues in  $F_1$  of  $X$  and  $Y$  in  $F$ , are not unique, since  $X_1Y_1$  is *any* line in  $F_1$  which is parallel to  $B_1C_1$ . We can take note of this difference by saying that points such as  $A, B, C, P, Q, R$ , and  $G$  are “specific parts” of  $\Delta$ , while  $X$  and  $Y$  are not specific.

Thus a specific part of  $\Delta$  is a point, line or other geometrical entity related to  $\Delta$  by a unique construction. The relation between  $\Delta$  and a specific part resembles the relation, in analysis, between the independent variable and a function of the variable; and it is often convenient to regard a specific part of  $\Delta$  as a geometrical function of  $\Delta$ .

**3. Geometrical function signs.** The symbol  $\Delta$  and the symbol for a specific part, such as  $G$ , may be regarded in the first instance as resembling proper names, though they are more closely analogous to names such as “Everyman” or “Tommy Atkins” than to names of actual individuals. To elucidate the relation of  $G$  to  $\Delta$ , consider the symbolic quotient  $G/\Delta$ ; this is a geometrical function sign meaning “the centroid of”.

Using  $G/\Delta$  as a function sign we may denote the centroid  $G_1$  of  $\Delta_1$  by  $\frac{G}{\Delta}(\Delta_1)$  and write  $G_1 = \frac{G}{\Delta}(\Delta_1)$ . It will be convenient to systematise this notation; thus if  $P$  is a specific part of  $\Delta$ , the isologous part of  $\Delta(X)$  will be denoted by the symbol  $P(X)$ ; that is,  $P(X)$  is defined to be  $\frac{P}{\Delta}(\Delta(X))$ .

There are two geometrical functions of two variables which we shall have occasion to use. If two triangles  $\Delta_1$  and  $\Delta_2$  are in perspective the centre is

denoted by  $\Pi(\Delta_1, \Delta_2)$  or, more concisely, by  $\Pi_{1,2}$ . If the triangles are homothetic the  $\Pi$  is replaced by a  $\Sigma$ , and the centre of similitude of  $\Delta_1$  and  $\Delta_2$  is  $\Sigma_{1,2}$ .

**4. Specific triangles.** A specific triangle of  $\Delta$  is one whose vertices and sides are specific parts of  $\Delta$ . It will usually be convenient to denote such a triangle by the symbol  $\Delta$  with a particular index, subscript or superscript. For example, the symbol  $\Delta_G$  denotes the medial triangle, whose vertices are the midpoints of the sides of  $\Delta$ . The  $A/\Delta$  vertex of this triangle is  $\frac{A}{\Delta}(\Delta_G)$ , which is  $A_G$ . This is the midpoint of  $BC$ .

Here are some other illustrations of the notation :

the centroid of  $\Delta_G$  is the centroid of  $\Delta$  ; thus  $G_G = G$  ;

the orthocentre of  $\Delta_G$  is the circumcentre of  $\Delta$  ; this can be written as  $H_G = O$ , if  $H$  is a proper symbol for the orthocentre of  $\Delta$  and  $O$  for the circumcentre ;

$\Delta$  and  $\Delta_G$  are homothetic with  $G$  as centre of similitude ; symbolically  $\Sigma_G = G$ .

The symbolism used for the medial triangle is a particular instance of this rule : if  $P$  is a specific point of  $\Delta$  then the specific triangle whose vertices are the intersections of  $AP$ ,  $BP$ ,  $CP$  with the opposite sides of  $\Delta$  is denoted by the symbol  $\Delta_P$ . Nearly all the specific triangles considered here are related to  $\Delta$  by this or the converse construction, and we need not now be concerned with the notation for other constructions. Another symbol formed according to this rule is  $\Delta_H$ , for the orthocentric, or pedal, triangle.

If we use  $\odot$  as a proper symbol for the circumcircle of  $\Delta$ , then from the nine-points circle theorem we have  $\odot_G = \odot_H$ . If the nine-points centre is  $N$  then  $N = O_G = O_H$ .

It is sometimes convenient to use a zero suffix, which does not modify the meaning of the symbol to which it is attached. Thus  $\Delta_0 = \Delta$ , and  $I_0 = I$ , etc.  $\Delta_0$  is a specific triangle which is identical with the original triangle.

In the equation  $\frac{P}{\Delta}(\Delta_Y) = P_Y$ , let  $P$  be a specific triangle  $\Delta_X$  of  $\Delta$ . Then  $\frac{\Delta_X}{\Delta}(\Delta_Y) = (\Delta_X)_Y$  ; let us write this as  $\Delta_{XY}$ . Thus  $\Delta_{XY}$  is the  $\Delta_X/\Delta$  part of  $\Delta_Y$ . For example,  $\Delta_{HG}$  is the orthocentric triangle of the medial triangle of  $\Delta$ . We can see also that  $P_{XY}$  denotes the  $P/\Delta$  part of  $\Delta_{XY}$ , or, what is the same thing, the  $P_X/\Delta$  part of  $\Delta_Y$ .

**5. Anti-triangles.** Let  $\Delta_X$  be a specific triangle of  $\Delta$ . Let us suppose that every triangle can be obtained from some other triangle by an application of the  $\Delta_X/\Delta$  construction. Then a triangle from which  $\Delta$  can be obtained in this way may be called an anti- $\Delta_X$  triangle of  $\Delta$ . To denote it the symbol is  $\Delta^X$ . Thus the function signs  $\Delta^X/\Delta$  and  $\Delta/\Delta_X$  are equivalent by definition. Since  $\frac{\Delta_X}{\Delta}(\Delta^X) = \Delta$  we have  $\Delta_X^X = \Delta$ .

If the anti- $\Delta_X$  triangle of a triangle is unique we also know that  $\frac{\Delta^X}{\Delta}(\Delta_X) = \Delta$ , or  $\Delta_X^X = \Delta$ .  $\Delta^G$  is the anti-medial triangle of  $\Delta$ . It is unique and is formed by drawing lines through  $A$ ,  $B$ ,  $C$  parallel to  $BC$ ,  $CA$ ,  $AB$ . There are four triangles, formed by any three points out of the incentre and ex-centres of  $\Delta$ , which have  $A$ ,  $B$ ,  $C$  as vertices of their orthocentric triangles. Hence the symbol  $\Delta^H$  is not as simple to interpret as  $\Delta^G$ . We return to this matter later, in Section 13.

**6. Manipulation of the indices.** Let  $\Delta_X$  be a specific triangle of  $\Delta$  and let

the  $P/\Delta$  part of  $\Delta_X$  be the  $Q/\Delta$  part of  $\Delta$ . This is shown by writing  $P_X = Q$ . We may assert the isologous relation for any other triangle  $\Delta_Y$ . Hence  $(P_X)_Y = Q_Y$ , or  $P_{XY} = Q_Y$ . This is the fundamental step in the manipulation of the indices.

Suppose  $\Delta_Y$  is also a specific triangle of  $\Delta$  and its  $Q/\Delta$  part is the  $R/\Delta$  part of  $\Delta$ . Then we may write  $Q_Y = R$ . Hence  $P_{XY} = R$ , since each is  $Q_Y$ . Thus in the equation  $Q_Y = R$  we can substitute for  $Q$  from the equation  $P_X = Q$ .

Let us also note the following: from  $P_X = Q$  we obtain, by isology in  $\Delta_X$ ,  $P_X^X = Q^X$ , or  $P = Q^X$ . Thus, for example, from  $G = G_G$  follows  $G^G = G$ , from  $N = O_G$  follows  $N^G = O$ , and from  $O = H_G$  follows  $O^G = H$ . Also from  $N = O_G$  and  $O = H_G$  follows  $N = H_{GG}$ , and, using  $N = O_H$ , we have  $N = H_{GH}$ . These are given as examples in manipulation, not as theorems of any great significance in themselves.

The symbols  $\Pi$  and  $\Sigma$  used with indices can be manipulated in analogous ways. Thus from  $P = \Pi_{X,Y}$  follows  $P_Z = \Pi_{XZ,YZ}$  (supposing  $\Delta_X$  and  $\Delta_Y$  are specific triangles of  $\Delta$ ). Similarly from  $P = \Sigma_{X,Y}$  follows  $P_Z = \Sigma_{XZ,YZ}$ . Another fundamental manipulation is this: if  $P$  is the homothetic centre of  $\Delta_X$  and  $\Delta_Y$ , then it is also the homothetic centre of their  $\Delta_Z/\Delta$  parts. Hence from  $P = \Sigma_{X,Y}$  follows  $P = \Sigma_{ZX,ZY}$ . For example, from  $\Sigma_G = G$  we obtain  $\Sigma^G, G^G = G^G$ , that is,  $\Sigma, G = G$ . Also from  $\Sigma_G = G$  we obtain  $\Sigma_H, HG = G$ .

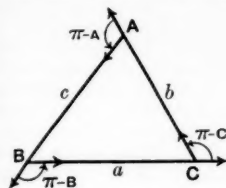
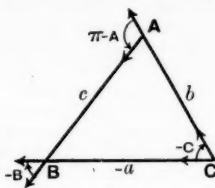
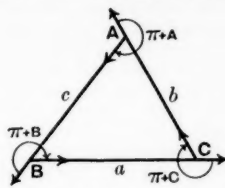
7. **Relations to be used in definitions.** The difficulty about the interpretation of the symbol  $\Delta^H$  is typical of many others that arise in the theory of the general triangle. It is related to the interpretation of the equation  $I_H = H$ , which appears to be untrue when  $\Delta$  is obtuse-angled. These difficulties can be resolved by limiting the relations which may be used in defining specific parts of  $\Delta$  to those of incidence, magnitude and order (linear or cyclical), and by defining some geometrical, trigonometrical and vector elements for  $\Delta$  to which these relations may refer. From this point of view it appears that the "incentre of the orthocentric triangle" is not a legitimate definition for a specific part of  $\Delta$ , and another interpretation for the symbol  $I_H$  is found.

8. **Definitions by incidence.** An incidence relation is primarily one between point and line (straight or curved) in that the point lies on or does not lie on the line. Other incidence relations are those of tangency, collinearity, or concurrency. Let us postulate that the line through any two points can be drawn, and that a circle with a given centre can be drawn to pass through a given point. Using these two constructions and incidence relations a large number of specific parts of  $\Delta$  can be derived from the geometrical elements of  $\Delta$  (that is, its vertices and sides) without appealing to linear or cyclical order. For example, we can bisect the sides at right angles and so obtain the circumcentre  $O$ . Specific parts such as this are "of the first kind", let us say. But we cannot distinguish between the internal and external bisectors of an angle without appealing to some relations of cyclical or linear order. Hence the incentre  $I$  is not definable by incidence relations alone, and is therefore "of the second kind". Evidently  $G, H, N$  and  $\odot$  are of the first kind. The reason for the distinction between the two kinds of specific parts will appear in Section 11.

9. **The vector circuit.** We turn now to relations of magnitude and order. As a basis of reference for these it is convenient to define a vector circuit in connection with  $\Delta$ .

Imagine the perimeter of the triangle as a motor track and suppose that a car, starting at  $A$ , pointing in the direction  $AB$ , completes a circuit of the perimeter, ending in the position from which it started. Then the length of each side of the triangle is obtainable from the milometer readings, and the

angles of the triangle can be obtained if we suppose the car to contain a direction indicator (compass needle). The angles turned through at each


 FIG. 1.  $\Delta$ .

 FIG. 2.  $\Delta_a$ .

 FIG. 3.  $\Delta_\delta$ .

corner are directly measurable and the supplements (mod  $2\pi$ ) of these angles give the interior angles of the triangle. In this way the trigonometrical elements— $a, b, c, A, B, C$ —of  $\Delta$  are related to the geometrical elements.

10. **Improper triangles.** Now we can imagine the car to travel backwards along any of the sides of the triangle. If it does this along  $BC$  and goes forwards along  $AB$  and  $CA$  we shall obtain  $\pi - A, -B, -C$  as the exterior angles. Hence  $A, \pi + B, \pi + C$  are the interior angles and  $-a, b, c$  are the lengths of the sides. We may notice that these trigonometrical elements ( $-a, b, c, A, \pi + B, \pi + C$ ) verify the sine and cosine formulae for a triangle. Let us call the geometrical figure  $ABC$  together with these trigonometrical elements an "improper triangle" and use the symbol  $\Delta_a$  to denote it (Fig. 2).

We have supposed so far that the positive direction for the measurement of angles is  $ABC$ . If, however,  $ACB$  is taken as the positive direction and the vector circuit is taken in the order  $ABC$ , then the exterior angles come to  $\pi + A, \pi + B, \pi + C$ , and hence the interior angles are  $-A, -B, -C$ . The geometrical elements together with the trigonometrical elements  $a, b, c, -A, -B, -C$  constitute the improper triangle  $\Delta_\delta$  (Fig. 3). ( $\delta$  is here read "deasil"; in mathematical diagrams the usual positive cyclical direction is "widdershins".)

Other improper triangles are definable. Thus the trigonometrical elements of  $\Delta_{bc}$  are  $a, -b, -c, A, \pi + B, \pi + C$ ; of  $\Delta_{a\delta}$  they are  $-a, b, c, -A, \pi - B, \pi - C$ , and so on. It can be seen that the suffixes  $a, b, c, \delta$  are commutative, at least for present purposes, so that  $\Delta_{bc} = \Delta_{cb}$ , etc., and that for each of the symbols,  $\Delta_{xx} = \Delta$  or  $\Delta_x = \Delta^x$ . Thus 16 different sets of trigonometrical elements can be associated with the geometrical figure. But it appears that the distinction between  $\Delta_{bc}$  and  $\Delta_a$  can be ignored in the defining of such specific parts as we shall be considering, and the number of different sets of elements is reduced to 8, of which 4 prove to be of particular interest. They are those associated with the symbols  $\Delta, \Delta_{a\delta}, \Delta_{b\delta}$  and  $\Delta_{c\delta}$ .

To abbreviate the three latter let us write  $\Delta_a, \Delta_b$  and  $\Delta_c$  for them. Thus we have this table of trigonometrical elements:

|            |             |                         |                      |
|------------|-------------|-------------------------|----------------------|
| $\Delta$   | $a, b, c,$  | $A, B, C,$              | cyclically positive; |
| $\Delta_a$ | $-a, b, c,$ | $-A, \pi - B, \pi - C,$ | cyclically negative; |
| $\Delta_b$ | $a, -b, c,$ | $\pi - A, -B, \pi - C,$ | cyclically negative; |
| $\Delta_c$ | $a, b, -c,$ | $\pi - A, \pi - B, -C,$ | cyclically negative. |

The geometrical elements collectively form the "geometrical" triangle, with symbol  $\nabla$  (del). The "complete triangle"  $\Delta$  consists of  $\nabla$  together with the vector circuit and the related trigonometrical elements. Then  $\Delta_a, \Delta_b, \Delta_c$ , etc., are improper triangles having the same  $\nabla$  as  $\Delta$ .

11. **Canonical definitions.** Let us now say that a specific part of  $\Delta$  is

canonical if defined by relations of incidence, magnitude or order which refer only to absolutes, to  $\mathcal{V}$ , to the vector circuit, to the trigonometrical elements, or to specific parts already defined. A specific part of the first kind, which is definable by incidence alone, can refer only to  $\mathcal{V}$ , not to the vector circuit or trigonometrical elements. Hence such a specific part of  $\Delta$  is identical with the isologous part of  $\Delta_a, \Delta_b, \Delta_c$ , etc. With a specific part of the second kind, however, the case is otherwise, and since  $\Delta$  and  $\Delta_a$ , for example, differ in their vector circuits and trigonometrical elements, isologous parts of these triangles are not identical. The analytical relations suggest the term "even" for specific parts of the first kind, and "not-even" for those of the second kind. Thus  $O$ , the circumcentre, definable by incidence alone, is even, and  $O_a = O_b = O_c = O$ .

12.  $I_a, I_b$  and  $I_c$ . To interpret the symbols  $I_a, I_b, I_c$ , and so on, we must first examine the canonical definition of  $I$ .  $AI$  is a line at an angle  $\frac{1}{2}A$  measured from  $AB$  towards  $AC$ ;  $BI$  is at an angle  $\frac{1}{2}B$  from  $BC$ ;  $I$  is the intersection of these lines. Hence for  $I_a$ :  $AI_a$  is at an angle  $\frac{1}{2}A_a$  (that is,  $\frac{1}{2}A$ ) from  $AB$ ;  $BI_a$  is at an angle  $\frac{1}{2}B_a$  (that is,  $\frac{1}{2}B + \frac{1}{2}\pi$ ) from  $BC$ . Hence  $AI_a$  is the same line as  $AI$ , and  $BI_a$  is the external bisector of the angle  $B$  of the triangle. Thus  $I_a$  is the excentre opposite  $A$ . It may be seen that  $I_b = I_c$ , and that  $I_a = I_a$ . Thus the excentres of  $\Delta$  are  $I_x$  for  $x = a, b, c$ , or  $a, b, c$ .

13.  $\Delta_H$  and  $\Delta^H$ . A complete triangle is not uniquely determined by its geometrical elements; hence the canonical definition of a specific triangle of  $\Delta$  requires more than the definition of its vertices. No difficulty arises in the case of  $\Delta_G$ ;  $\mathcal{V}_G$  and  $\mathcal{V}$  are homothetic and we assign to  $\Delta_G$  the same angles as to  $\Delta$ .  $\Delta_G$  is the proper triangle,  $\Delta_{Gx}$  is the same as  $\Delta_{Gx}$  for  $x = a, b$ , etc., and  $P_{Gx} = P_{Gx}$ . Similar relations hold for  $\Delta^G$ .

Let us now examine the meaning of the symbol  $\Delta_H$ . If  $\Delta$  is not obtuse-angled the angles of the triangle  $\Delta_H$  are given by  $A_H = \pi - 2A, B_H = \pi - 2B, C_H = \pi - 2C$ . But if  $A$  is obtuse, the angles of the proper triangle are given by  $A' = 2A - \pi, B' = 2B, C' = 2C$ . These are consistent with the formulae for  $A_H, B_H, C_H$  if  $A_H = -A', B_H = \pi - B', C_H = \pi - C'$ . Thus  $A_H, B_H, C_H$  are the angles of an improper triangle of the type  $\Delta_a$ . Let us write  $\Delta_{|H|}$  for the proper triangle, then  $\Delta_{a|H|}$  represents the improper triangle we are concerned with. Similarly the angles  $\pi - 2A, \pi - 2B, \pi - 2C$  refer to  $\Delta_{b|H|}$  if  $B$  is obtuse and to  $\Delta_{c|H|}$  if  $C$  is obtuse. Hence we define  $\Delta_H$  to mean  $\Delta_{x|H|}$  when  $x = 0, a, b, c$ , according to the shape of  $\Delta$ ; it is a proper triangle only if  $\Delta$  has no acute angle. We say therefore that

$I_H = H$  means  $I_{|H|} = H$  if  $\Delta$  is not obtuse-angled,

$I_{a|H|} = H$  if  $A$  is obtuse, and so on;

and it is seen that the result is verified in all four cases.

Consider now the symbol  $\Delta^H$ . This must denote a triangle whose orthocentric triangle is  $\Delta$ , a proper triangle. Hence  $\Delta^H$  is not obtuse-angled; and it is therefore the excentric triangle. Now  $\Delta^H$  has vertices  $I_a, I_b, I_c$  with angles  $\frac{1}{2}\pi - \frac{1}{2}A, \frac{1}{2}\pi - \frac{1}{2}B, \frac{1}{2}\pi - \frac{1}{2}C$ . Hence  $\Delta^H_a$  has vertices  $(I_a I_b I_c)_a$ , with angles  $\frac{1}{2}\pi - \frac{1}{2}A_a, \frac{1}{2}\pi - \frac{1}{2}B_a, \frac{1}{2}\pi - \frac{1}{2}C_a$ , that is, vertices  $I I_c I_b$  and angles  $\frac{1}{2}\pi + \frac{1}{2}A, \frac{1}{2}B, \frac{1}{2}C$ . Thus  $\Delta^H_a$  is the proper triangle with vertices  $I I_c I_b$ .

Since many cases arise in which the form of statement is dependent on the shape of  $\Delta$ , it is convenient to make a classification of triangles thus:

$\Delta_X$  is in class 0 if  $\mathcal{V}_X$  has no obtuse angle;

in class  $a$  if  $A_X$  is obtuse;

in class  $b$  if  $B_X$  is obtuse;

in class  $c$  if  $C_X$  is obtuse.

If we use  $\mathfrak{C}$  as the class symbol for  $\Delta$  and  $\mathfrak{C}_X$  as the class symbol for  $\Delta_X$  we may summarise the properties of  $\Delta_H$  and  $\Delta^H$  just discussed as follows :

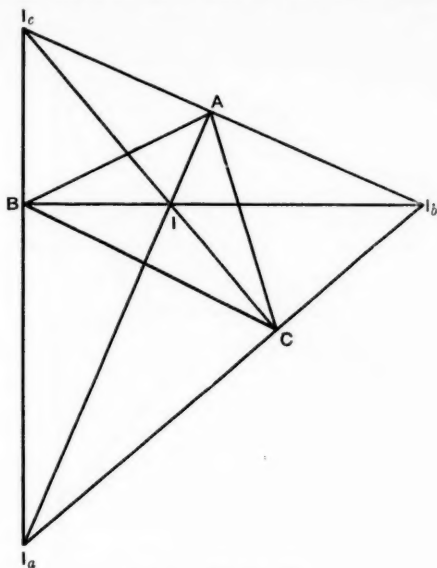


FIG. 4.  $\Delta_H$  and  $\Delta^H$ .

$$I = HH = AH_a; I_a = AH = HH_a; I_b = BH = CH_b; I_c = CH = BH_c.$$

- (i)  $\Delta_H = \Delta_{x|H|}$  for  $x = \mathfrak{C}$ ;
- (ii) the vertices of  $\Delta^H_x$  are  $II_aI_bI_c$  with the omission of  $I_x$  for  $x = 0, a, b, c$ ;
- (iii)  $\mathfrak{C}_x = x$  for  $x = 0, a, b, c$ ;
- (iv)  $I_H = H$ , meaning  $I_{x|H|} = H$  for  $x = \mathfrak{C}$ ;
- (v)  $I_x = HH_x$  for  $x = 0, a, b, c$ .

15.  $\Delta^T$  and  $\Delta_T$ . We consider now some other specific triangles of  $\Delta$ . The symbol  $\Delta^T$  denotes the triangle formed by the tangents at  $A, B, C$  to the circumcircle, the circumtriangle, say. Now the sides of  $\Delta^T$  are parallel to the sides of  $\Delta_H$  (for example, if  $\Delta$  is acute-angled,  $\angle C^TAB = \angle ACB$  in the alternate segment  $= \angle AC_HB_H$  from the cyclic quadrilateral  $BC_HB_HC$ ). Hence the angle formulae used for  $\Delta_H$  also apply to  $\Delta^T$ . For the same reasons as in the case of  $\Delta_H$  we write  $\Delta^T = \Delta_x^T$  where  $x = \mathfrak{C}$ ; and  $\Delta^T$  is proper only if  $\Delta$  is acute-angled.

The symbol  $\Delta_T$  denotes a triangle of which  $\Delta$  is the circumtriangle. Since  $\Delta$  is proper,  $\Delta_T$  is acute-angled. Now  $\nabla$  forms the geometrical elements of the circumtriangle of four triangles, these four being those whose vertices are the points of contact of the sides of  $\Delta$  with its inscribed and escribed circles. Of these only the triangle related to the inscribed circle is acute-angled; hence the symbol  $\Delta_T$  applies to this "in-triangle".

We can see that  $\Delta_{T_a}$  is the proper triangle whose vertices are the points of contact of the sides of  $\Delta$  with the escribed circle opposite to  $A$ , an ex-triangle of  $\Delta$ . This triangle is obtuse-angled at  $A_{T_a}$ . The circumcentre of this triangle is  $I_a$ ; thus  $I_a = O_{T_a}$ .

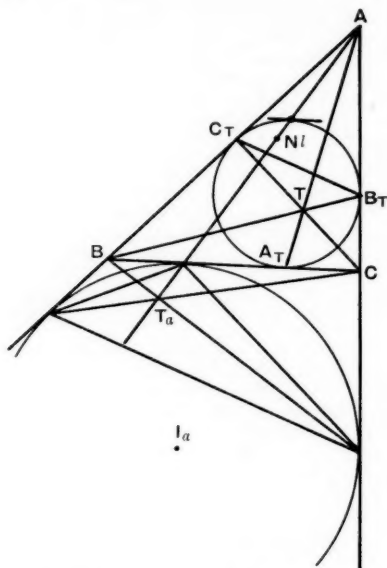


FIG. 5.  $\Delta_T$  and  $\Delta_{T_a}$ .

$I_a = O_{T_a}$ ; circle centre  $I_a$  touches  $BC$ ,  $CA$ ,  $AB$  at  $A_{T_a}$ ,  $B_{T_a}$ ,  $C_{T_a}$  respectively; the tangent to the inscribed circle parallel to  $BC$  has  $AOT$  as point of contact.

Properties of  $\Delta^T$  and  $\Delta_T$  analogous to those of  $\Delta_H$  and  $\Delta^H$  are:

- (i)  $\Delta^T = \Delta_x |T|$ , where  $x = \mathfrak{C}$ ;
- (ii) the vertices of  $\Delta_{T_x}$  are the points of contact with the sides of  $\Delta$  of a circle centre  $I_x$  for  $x = 0, a, b, c$ ;
- (iii)  $\mathfrak{C}_{T_x} = x$  for  $x = 0, a, b, c$ ;
- (iv)  $I^T = O'$ , meaning  $I_x |T| = O$  for  $x = \mathfrak{C}$ ;
- (v)  $I_x = O_{T_x}$  for  $x = 0, a, b, c$ .

15. Some homothetic centres. Much of the specific geometry of  $\Delta$  which follows is dependent on properties of homothetic triangles, for we can see that  $\Delta$  is homothetic with the following specific triangles:

$\Delta_G$ ; this symbol has one suffix;

$\Delta_{TH}$  and  $\Delta_{HT}$ ; these symbols have two suffixes;

$\Delta_{PQR}$ , where  $PQR$  is a permutation of  $THG$  or of subscripts  $T, H$  and superscript  $G$ ; these symbols have three suffixes.

Let us say that the centres are of the first, second and third rank respec-

tively. The single homothetic centre of the first rank is the well-known point  $G$ . Some of the centres of the third rank are also well known; they are identified in Sections 18, *et seq.*

Let us now consider the second rank centres.

16.  $J$  and  $V$ . We have seen that  $\Delta^T$  and  $\Delta_H$  are homothetic. Let  $J$  denote the homothetic centre. Then

$$J = \Sigma^T, H = \Sigma^T, TH = \Sigma, TH.$$

By isology in  $\Delta_T$ ,

$$J_T = \Sigma^T, HT = \Sigma, HT.$$

Hence

$$J_T = \Sigma^H, HT = \Sigma^H, T$$

Thus  $\Delta^H$  and  $\Delta_T$  are homothetic; this is otherwise evident since  $a^H$  and  $a_T$  are parallel, each being perpendicular to  $AI$ . Using  $V$  as a proper symbol for this centre of similitude, we have  $V = J_T$ . We have also

$$JH = \Sigma^TH, H^H = \Sigma^TH, = \Sigma^H, T = V.$$

There are four  $V$  points given by  $J_{Tx} = V_x = \Sigma^H, Tx$  for  $x = 0, a, b, c$ .

To summarise these relations:

$$J = \Sigma^T, H = \Sigma, TH.$$

$$V_x = \Sigma^H, Tx = \Sigma, HTx, \text{ for } x = 0, a, b, c.$$

$$J = V_H = V^T, \text{ that is, } J = V_x | H | = V_x | T | \text{ for } x = \mathbb{C}.$$

$$V = J^H = J_T, \text{ or } V_x = J^H_x = J_{Tx} \text{ for } x = 0, a, b, c.$$

17.  $J$  and  $O^T$  are on the Euler line. From a fundamental theorem about homothetic triangles we can see that  $IT, I_H$  and  $\Sigma^T, H$ —that is,  $O, H$  and  $J$ —are collinear. Thus  $J$  is on the Euler line of  $\Delta$ . From the same theorem it follows that  $O^T, O_H$  and  $J$ —that is,  $O^T, N$  and  $J$ —are collinear.  $O^T$  also is therefore on the Euler line.

Since  $O^T, O$  and  $J$  are collinear it follows by isology in  $\Delta_T$  that  $O, O_T$  and  $J_T$ —that is,  $O, I$  and  $V$ —are collinear. Thus  $V$  is on the line  $OI$ .

18. The Lemoine point. One of the third rank homothetic centres is the Lemoine point  $K$ . This is the point of concurrence of  $AA^T, BB^T, CC^T$ . That these lines do in fact concur may be seen by Ceva's theorem, since

$$ATC \cdot BTA \cdot CTB = CBT \cdot ACT \cdot BAT.$$

Thus  $K = \Pi, T$ ; and  $K$  is even.

A well-known and important property of  $K$  is its relation with  $G$ , shown as follows:  $B, C_H, B_H, C$  lie on a circle, diameter  $BC$ , centre  $A_G$ . Hence  $A_G B_H C$  is isosceles, and  $\angle A_G B_H C = C = \angle A C_H B_H$ . Hence  $A_G B_H$  is the tangent at  $B_H$  to the circle  $A B_H C_H$  and similarly  $A_G C_H$  is the tangent at  $C_H$  to this circle. The triangles  $A B_H C_H$  and  $A B C$  are inversely similar;  $A_G$  and  $A^T$  are corresponding points relative to the two triangles; hence (Fig. 6)

$$\angle C_H A A_G = \angle C A A^T,$$

that is,  $\angle B A G = \angle C A K$ . Similarly  $\angle A C G = \angle B C K$  and  $\angle C B G = \angle A B K$ .  $K$  is the "isogonal conjugate" of  $G$  and we write  $K = \hat{G}$ . Let  $AK$  cut  $B_H C_H$  at  $X$ . Then, since  $\angle B_H A X = \angle B A A_G$ ,  $X$  and  $A_G$  are corresponding points in the similar figures. Hence  $X$  is the midpoint of  $B_H C_H$  and is therefore  $A_{GH}$ . Thus  $A^T A_{GH}$  goes through  $K$ ; and similarly for  $B^T B_{GH}$  and  $C^T C_{GH}$ .

We write  $K = \Pi^T, GH$ . But we know that  $\Delta^T$  and  $\Delta_{GH}$  are homothetic, since each is homothetic with  $\Delta_H$ . Hence  $K = \Sigma^T, GH$ , which is also  $\Sigma, TGH$ .

19. The points  $T$  and  $U$ . If we write  $\Delta_T$  for  $\Delta$  in  $K = \Pi, T$ , we obtain  $K_T = \Pi_T$ . Thus  $K_T$  is the centre of perspective of  $\Delta$  and  $\Delta_T$ . We use  $T$  as

a proper symbol for this point (the Gergonne point). Since  $K = \Sigma^{T, GH}$ , isology in  $\Delta_T$  gives  $K_T = T = \Sigma_{,GHT}$ . To denote the Gergonne point of the medial triangle let us use the symbol  $U$ ; that is,  $U = T_G$ . Since

$$T = \Sigma_{,GHT} = \Sigma^G_{,HT},$$

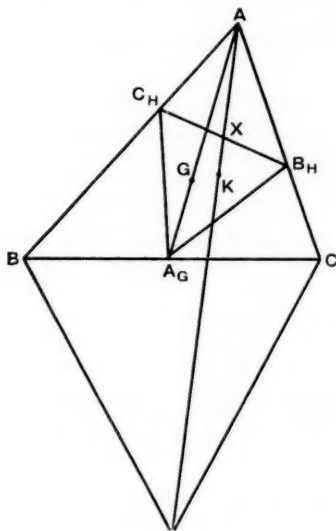


FIG. 6.

$$K = \hat{G} = \Sigma^{T, GH}, \quad X = A_{GH}, \quad AAT, \text{ etc., concurrent at } K.$$

we have  $U = T_G = \Sigma_{,HTG}$  which is  $\Sigma^H_{,TG}$ . Thus  $K$ ,  $T$  and  $U$  are three of the third rank homothetic centres.  $T$  and  $U$  are not-even parts of  $\Delta$ . Thus there are four  $T$  points and four  $U$  points :

$$T_x = \Pi_{T_x} = \Sigma_{x,GHT_x} \quad \text{for } x = 0, a, b, c;$$

$$U_x = T_{G_x} = T_{xG} = \Sigma_{x,HT_xG} \quad \text{for } x = 0, a, b, c.$$

Since  $K_{TG}$  and  $K^H$  are corresponding points in the homothetic triangles  $\Delta_{TG}$  and  $\Delta^H$ , whose centre of similitude is  $U$ , and since  $K_{TG} = T_G = U$ , we have the result  $K^H = U$  or  $K = U_H$ .

20. Relations between  $O$ ,  $H$  and  $I$ . We have noticed the equations

$$O = H_G, \quad H = I_H, \quad I = O_T, \quad \dots\dots\dots(i) \quad (ii) \quad (iii)$$

From (i) and (ii),  $O = I_{HG}$ , and therefore from (iii)  $O = O_{THG}$ . Hence  $\Delta$  and  $\Delta_{THG}$  have a common  $O/\Delta$  point at  $O$ . This therefore is their centre of similitude. Hence  $O = \Sigma_{,THG}$ .

Writing  $\Delta_T$  for  $\Delta$  in  $O = \Sigma^T_{,HG}$  gives  $O_T = \Sigma_{,HGT}$  or  $I = \Sigma_{,HGT}$ .

Writing  $\Delta_H$  for  $\Delta$  in  $I = \Sigma^H_{,GT}$  gives  $I_H = \Sigma_{,GTH}$  or  $H = \Sigma_{,GTH}$ .

Thus  $O$ ,  $H$  and  $I$  are three other third rank centres of similitude.

21. **Similitude.** It is interesting to see the results tabulated thus :

|                     |                     |
|---------------------|---------------------|
| $O = H_G$           | $U = T_G$           |
| $H = I_H$           | $T = K_T$           |
| $K = U_H$           | $I = O_T$           |
| $J = V_H$           | $V = J_T$           |
| $O = \Sigma^T_{HG}$ | $U = \Sigma^H_{TG}$ |
| $H = \Sigma^{TG}_H$ | $T = \Sigma^{HG}_T$ |
| $K = \Sigma^T_{GH}$ | $I = \Sigma^H_{GT}$ |
| $J = \Sigma^T_{HJ}$ | $V = \Sigma^H_{JP}$ |

(Here the subjects of the equations in the left-hand column are all even ; of those in the other column, not-even.)

There are obvious analogies between the two columns of results, which provide an example of a relationship which is here called "similitude". In projective geometry we may have a figure  $F_1$  in which a property  $\theta_1$  of a triangle  $\Delta_1$  is exemplified. By projection a figure  $F_2$  containing a triangle  $\Delta_2$  is obtained, and the property  $\phi_2$  of  $\Delta_2$  follows as a consequence of  $\theta_1$  of  $\Delta_1$ . But here we are interested, not so much in the relationship between  $\theta_1$  and  $\phi_2$  as in that between the isologous properties,  $\theta$  and  $\phi$ , of  $\Delta$ . This is the relationship "similitude", a compound of isology and projection, or of isology and homography.

The essential properties of a similitude are :

- (i) it is a 1-1 correspondence between two sets of specific geometrical entities, whereby point corresponds to point, line to line, and curve to curve of like degree ;
- (ii) for every incidence property of the first set of entities there is a similar incidence property of the second set.

Similitude differs from homography in that it does not necessarily provide a 1-1 correspondence for non-specific entities. Moreover, it may be proved that the cross-ratio of a range or pencil is not necessarily equal to that of the similitive range or pencil. However, if the original range is harmonic then so is the related range, and if two ranges are equi-cross then so are the related ranges.

22. **The d-similitude.** The particular similitude relating  $O$ ,  $H$ ,  $K$  and  $J$  to  $U$ ,  $T$ ,  $I$  and  $V$  arises as follows. We first define a conic  $\phi$  which touches the sides of  $\Delta^H$  at the vertices of  $\Delta$ . (That a conic can be drawn to satisfy these six conditions may be seen by Pascal's theorem.) Let  $F_{(1)}$  be a figure containing a triangle  $\Delta_{(1)}$  and the conic  $\phi_{(1)}$ . By projection we form a figure  $F_{(2)}$  in which  $\nabla_{(2)}$  is the projection of  $\nabla_{(1)}$  ; and  $\Delta_{(2)}$  is defined to be the proper triangle associated with  $\nabla_{(2)}$ . The projection is chosen so that the line at infinity in  $F_{(1)}$  projects to the line at infinity in  $F_{(2)}$ , and so that  $\phi_{(1)}$  projects to a circle, which must be  $\odot_{(2)}$ , the circumcircle of  $\Delta_{(2)}$  ; the projection is cylindrical.

Let  $P_{(1)}$ , a specific part of  $\Delta_{(1)}$  in  $F_{(1)}$ , project to  $P'$  in  $F_{(2)}$ . Let  $P'$  be the  $Q/\Delta$  part of  $\Delta_{(2)}$ , that is,  $P' = Q_{(2)}$ . Then the relation between  $P$  and  $Q$ , the specific parts of  $\Delta$ , is a similitude, and we shall write  $P = Q_d$ , or  $P^d = Q$ .

The tangents at  $A_{(1)}$ ,  $B_{(1)}$ ,  $C_{(1)}$  to  $\phi_{(1)}$  project to the tangents at  $A_{(2)}$ ,  $B_{(2)}$ ,  $C_{(2)}$  to  $\odot_{(2)}$ . Hence  $\nabla^H_{(1)}$  projects to  $\nabla^T_{(2)}$  and  $\nabla^H_d = \nabla^T$ . Since  $\nabla_{T(1)}$  is inscribed in  $\nabla_{(1)}$  and is homothetic with  $\nabla^H_{(1)}$  it projects to  $\nabla_{H(2)}$  which is inscribed in  $\nabla_{(2)}$  and is homothetic with  $\nabla^T_{(2)}$ . Thus  $\nabla^d_{T(1)} = \nabla_H$ .

Some other results obtainable in this way are :

$$\begin{aligned} \nabla_G^d &= \nabla_G, & \nabla^{Gd} &= \nabla^G, & P_G^d &= P_G^d, & P^{Gd} &= P^d G, \\ \nabla^{HGd} &= \nabla^{TG}, & \nabla_{GT}^d &= \nabla_{GH}, & \nabla^{GHd} &= \nabla^{GT}, & \nabla_{TG}^d &= \nabla_{HG}; \\ G^d &= G, & \text{since } (\Pi, G)^d &= \Pi, G; & I^d &= K, & \text{since } (\Pi, H)^d &= \Pi, T; \\ T^d &= H, & \text{since } (\Pi, T)^d &= \Pi, H; & U^d &= O, & \text{since } (\Sigma^{H, TG})^d &= \Sigma^{T, HG}; \\ V^d &= J, & \text{since } (\Sigma^{H, T})^d &= \Sigma^{T, H}. \end{aligned}$$

The similitude also leads to these properties :

$O$  is the centre of  $\odot$ , hence  $U$  is the centre of  $\phi$ .

Since  $O = \Pi^T, G$ ,  $U = (\Pi^T, G)_d = \Pi^{H, G}$ , that is,  $U$  is the point of concurrence of the lines joining the excentres to the midpoints of the sides of  $\Delta$ .

From the concurrence of  $I_a A_G$ ,  $I_b B_G$ ,  $I_c C_G$  at  $U$  we can obtain, by isology in  $\Delta_a$ , the concurrence of  $I A_G$ ,  $I_c B_G$ ,  $I_b C_G$  at  $U_a$ , which is the centre of  $\phi_a$  (Fig. 7).

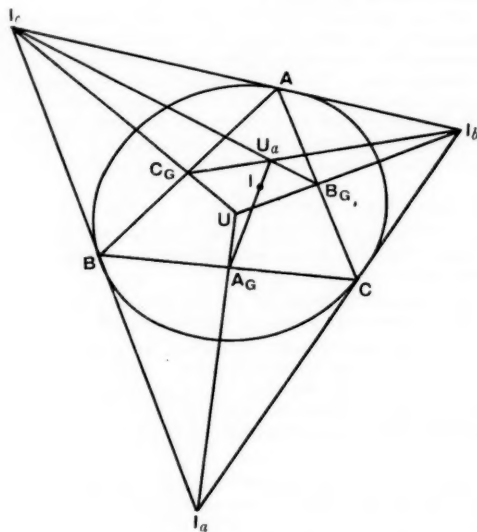


FIG. 7.  
 $\phi$ ,  $U$  and  $U_a$ .

Since  $O$ ,  $G$ ,  $H$  and  $J$  are collinear, their similitives  $U$ ,  $G$ ,  $T$  and  $V$  are also collinear.

We may notice here an example of an essential difference between a similitude and a homographic relation. This  $d$ -similitude provides a 1-1 correspondence between pairs of points on the median  $AG$  (since the line itself is self-corresponding). The number of self-corresponding points on the line is not two, but is quite indefinite; five of the points are  $A$ ,  $G$ ,  $A_G$ ,  $A^G$  and the point at infinity. Of course, there is also an indefinite number of points in the line which are not self-corresponding, for example, the points in which  $AG$  cuts  $\odot$  or  $\phi$  or the sides of any of the specific triangles considered in this article other than  $\Delta$ ,  $\Delta_G$  and  $\Delta^G$ .

Analysis shows that  $P^d$  may be obtained by substituting  $a^2, b^2, c^2$ , for  $a, b, c$  in the areal coordinates of  $P$ . Thus any point whose areal coordinates are independent of  $a, b, c$  is self-corresponding.

**23. The Nagel point.** Here is an example of the use of the symbolic method. The Nagel point, symbol  $Nl$ , is the point of concurrence of the lines joining the vertices of  $\Delta$  to the points of contact of the sides with the corresponding escribed circles, that is, the point of concurrence of  $AA_{Ta}, BB_{Tb}, CC_{Tc}$ , or of  $AT_a, BT_b, CT_c$ . A fundamental property of  $Nl$  is the fact that it is the in-centre of the anti-medial triangle; that is, that  $Nl = I^G$ . We can show that this rather obscure theorem is implicit in the very obvious relation that  $AT, A_G$  and  $O$  are collinear. For from this by the  $d$ -similarity,  $AH, A_G$  and  $U$  are collinear; that is,  $I_a, A_G$  and  $U$ . By isology in  $\Delta_a, I, A_G$  and  $U_a$  are collinear, and hence by isology in  $\Delta^G, I^G, A$  and  $U_a^G$ ; that is  $I^G, A$  and  $T_a$ . Similarly  $I^G$  is on  $BT_b$  and  $CT_c$ , and is  $Nl$ . There therefore are four Nagel points,  $Nl_x = I_x^G$ , for  $x = 0, a, b, c$ .

**24. Isogonal and isotomic conjugates.** Several properties of  $\Delta$  may be obtained by the use of isogonal or isotomic conjugates. Points  $P$  and  $Q$  are isogonal conjugates with respect to  $\Delta$  if  $\angle BAP = \angle QAC, \angle CBP = \angle QBA, \angle ACP = \angle QCB$ . To denote the relation between  $P$  and  $Q$  we write  $P = \hat{Q}$ , or  $P = \wedge Q$ . We have already seen that  $\hat{G} = K$ . Other well-known isogonal relations are expressible as  $\hat{O} = H$ , and  $\hat{I}_x = I_x$ , for  $x = 0, a, b, c$ .

If  $P$  and  $Q$  are isogonal conjugates with respect to  $\Delta_X$  we write  $P = \wedge_X Q$ , and the symbol  $\wedge_X Q_X$  as  $(\hat{Q})_X$ .

An important theorem about such isogonal conjugates is this: if a point  $P$  moves along a straight line  $l$ , then  $Q$  moves along a circumconic of  $\Delta$ . The proof can be obtained by Chasles' theorem. The locus of  $Q$  is the "isogonal point transform" of  $l$ , and can be denoted by  $\bar{l}$  with a circumflex. Here, however, we shall be concerned only with isogonal conjugate points.

Points  $P$  and  $Q$  are isotomic conjugates with respect to  $\Delta$  if the following length equalities hold:

$$BA_P = A_QC, \quad CB_P = B_QA, \quad AC_P = C_QB.$$

We shall write  $P = \bar{Q}$ . We know that  $G = \bar{G}$ ; that  $Nl = \bar{T}$  is readily seen, since  $BA_T = s - b = AT_aC$ , etc.

It may be proved for isotomic conjugates as for isogonal conjugates, that if  $P$  moves on a straight line  $l$  then  $\bar{P}$  moves on a locus  $\bar{l}$  which is a circumconic of  $\Delta$ .

If  $P = \bar{Q}$ , then it follows, from the properties of the cylindrical projection from which the  $d$ -similarity was obtained, that  $P^d = (\bar{Q}^d)$ ; but if  $P = \hat{Q}$ , then the relation between  $P^d$  and  $Q^d$  is more complicated.

**25. Examples involving specific isogonal and isotomic relations.** We turn now to some properties of  $\Delta$  involving isogonal and isotomic points.

The side  $a$  of  $\Delta$  cuts  $\odot_G$  at  $A_G$  and  $A_H$ . Hence, by isology in  $\Delta^G, a^G$  cuts  $\odot$  at  $A$  and  $A_H^G$ . Thus, by drawing lines through  $A, B, C$  parallel to  $a, b, c$ , to cut  $\odot$  again, the vertices of  $\Delta_H^G$  are obtained (Fig. 8). This triangle is homothetic with  $\Delta_H$  and hence with  $\Delta^T$ . Let  $\Sigma^T, H^G$  be  $O^*$ . (This is  $\Sigma_{TH}^G$ , one of the third rank homothetic centres with superscript  $G$ .) Then  $O^*$  lies on the line joining the  $O/\Delta$  points of  $\Delta^T$  and  $\Delta_H^G$ , that is, on the line  $OTO$ , which is the Euler line of  $\Delta$ . Now let  $AH$  cut  $\odot$  at  $A_H$ . This point is diametrically opposite  $A_H^G$  in  $\odot$ , and hence  $\Delta_H$  is congruent and homothetic with  $\Delta_H^G$ , and hence homothetic with  $\Delta^T$ . Let  $\Sigma^T, H$  be  $Gb$  (the Gob point;



Therefore, from (iv),

$$NI = \Lambda (Gb_T).$$

Similarly, from (ii), which is  $K = \Lambda^T O^*$ , we get, by isology in  $\Delta_T$ ,

$$K_T \text{ (or } T) = \Lambda (O^*_T).$$

Now from (i),  $OT$ ,  $O^*$  and  $Gb$  are collinear; hence, by isology in  $\Delta_T$ ,  $O$ ,  $O^*_T$  and  $Gb_T$  are collinear. The isogonal conjugates of these points, namely  $H$ ,  $T$  and  $NI$ , lie on a circumconic of  $\Delta$ . The isotomic conjugates of these, which are  $\bar{H}$ ,  $NI$ , and  $T$  respectively, lie on a straight line. Now  $\bar{T} = NI = I^G$ ; hence by the  $d$ -similarity,

$$\bar{H} = NI^d = I^{G^d} = I^{dG} = K^G.$$

Thus  $K^G$ ,  $NI$  and  $T$  lies on a straight line. By isology in  $\Delta_G$  the collinearity of  $K$ ,  $I$  and  $U$  follows.

#### 26. Miscellaneous examples.

- (i) If  $P = \hat{Q}$ , then  $P^d = \hat{\Lambda}(Q^d)$ .
- (ii)  $\hat{U} = V$ .
- (iii)  $\hat{J} = \bar{H}$ .
- (iv)  $H$ ,  $U$ ,  $I_G$  and  $O^H$  are collinear.
- (v) If  $Fh = \Sigma^{G_T, H_G}$  (the Feuerbach point) then  $FhT = FhG^d$ .
- (vi)  $Fh$ ,  $O^*_T$  and  $G$  are collinear.
- (vii)  $Fh$ ,  $H$  and  $Gb_T$  are collinear.
- (viii) There is an  $e$ -similarity which is symmetrical ( $P_e = P^e$ ) such that  $O_e = J$ ,  $G_e = H$ ,  $K_e = K$ ,  $\odot_e = \odot$ , and  $O^*_e = Gb$ .

R. H. C.

1747. It is fairly simple for the boys of Class 3A—they only take the engines to pieces. But in classes 4A and 5A are the "gen" boys—they put engines together and make them work. These boys have already become acquainted with that schoolboy's nightmare, the quadratic equation. Now they are applying mathematics with slide-rules and Vernier gauges and micrometers.—*Coventry Evening Telegraph*, July 18, 1952. [Per Mr. C. C. Puckette.]

1748. Have you read any of the works of Dr. Salmon? I have just finished his volume on Infallibility, which fills me with admiration of its easy movement, command of knowledge, singular felicity of disentanglement, and great skill and point in argument; though he does not quite make one love him.—Letter from W. E. Gladstone to Lord Acton, quoted by J. Morley, *Life of William Ewart Gladstone* (1906 edition), II, p. 657.

1749. *Duodecimals Infraordinary*. A popular writer . . . has observed that "why anyone should select the best hundred, more than the best eleven, or the best thirty books, it is hard to conjecture". . . . Indeed, if our arithmetical notation had been duodecimal instead of decimal, I should no doubt have made up the number to 120.—Introduction to each volume of the series of Sir John Lubbock's *Hundred Books* (circ. 1890). [Per Prof. E. H. Neville.]

## SOME SIMPLE GEOMETRICAL EXTREMAL PROBLEMS.

BY S. J. TAYLOR.

Suppose  $C$  is a convex curve bounding a plane domain of area  $A$ .  $l$  will denote the length of  $C$  and  $d$  its diameter; that is, the upper bound of the distance between any pair of its points. It is clear that, for all convex curves  $C$ ,  $2d \leq l \leq \pi d$ . In this paper I consider classes of convex curves  $C$  with a given value of  $l$  satisfying one or more further conditions. The aim will be in each case to find the curves which enclose minimum or maximum area. The existence of a curve of a given class  $\Sigma$  for which the area enclosed attains the maximum or minimum possible follows from compactness in the class of convex curves in a bounded part of the plane.

I am indebted to Prof. Besicovitch for suggesting the problems considered, and for his help and encouragement throughout.

1. *Convex curves lying in a closed disc.* Suppose  $Q$  is the circumference of a disc of radius 1. Let  $A, B$  be points of  $Q$  subtending an angle  $\theta$  at the centre.

Denote by  $|AB|$  the distance from  $A$  to  $B$  and by  $|\widehat{AB}|$  the length of the arc of  $Q$  joining  $A$  to  $B$ . Let

$$k = |\widehat{AB}| - |AB| = \theta - 2 \sin \frac{1}{2}\theta,$$

$$dk/d\theta = 1 - \cos \frac{1}{2}\theta. \dots\dots\dots(1)$$

so that

The area of the segment between the chord and the arc is given by

$$a = \frac{1}{2}(\theta - \sin \theta),$$

$$da/d\theta = \frac{1}{2}(1 - \cos \theta). \dots\dots\dots(2)$$

so that

From (1) and (2), it follows that

$$da/dk = 2 \cos^2 \frac{1}{2}\theta. \dots\dots\dots(3)$$

The relation (3) implies the following:

*Lemma.* The function  $da/dk$  is a strictly decreasing function of  $\theta$  for

$$0 < \theta < 2\pi.$$

We can now consider

*Problem 1.* Suppose  $0 < l \leq 2\pi r$ , and  $\Sigma$  is the class of convex curves  $C$ , of length  $l$ , lying entirely in a circular disc  $H$  of radius  $r$ . What curves of  $\Sigma$  enclose minimum area?

*Case (i)*  $0 < l \leq 4r$ : A segment of a straight line of length  $\frac{1}{2}l$  can be considered as a closed convex curve of length  $l$  if we think of it as described first in one direction, then in the other direction. Convex curves of this kind enclose zero area. However, for  $l \leq 4r$ , a segment of length  $\frac{1}{2}l$  can lie entirely in a disc of radius  $r$ . Thus the problem is solved trivially in this case.

*Case (ii)*  $4r < l \leq 2\pi r$ : Let  $A$  denote a curve of  $\Sigma$  which bounds a set of minimum area. As  $A$  is convex, its curvature exists at almost all points. We first of all prove

(a) *At no point of  $A$  not on  $Q$ , the circumference of  $H$ , can the curvature exist and be finite and positive.*

Suppose, if possible,  $P$  is a point on  $A$  at a distance  $\delta > 0$  from  $Q$  at which the curvature  $\kappa$  of  $A$  exists with  $0 < \kappa < \infty$ . Suppose  $PN$  is the inward normal to  $A$  at  $P$  of length  $\rho = 1/\kappa$  (see Fig. (i)). Let  $X, Y$  be points on  $A$  within  $\delta$  of  $P$  such that

$$\text{angle } XNP = \text{angle } PNY = \theta > 0.$$

Then, since the curvature exists at  $P$ ,

$$XN = \rho + o(\theta^2). \dots\dots\dots(4)$$

Also

$$|\widehat{XP}| = \rho\theta + o(\theta^3) \quad \dots\dots\dots(5)$$

and the area of the segment bounded by the chord and arc  $XP$  is given by

$$a_1 = \theta^3 \left[ \frac{\rho^2}{12} + o(1) \right],$$

and so there exists a  $\theta_1 > 0$  such that, for  $0 < \theta < \theta_1$ ,

$$a_1 > \rho^2\theta^3/15. \quad \dots\dots\dots(6)$$

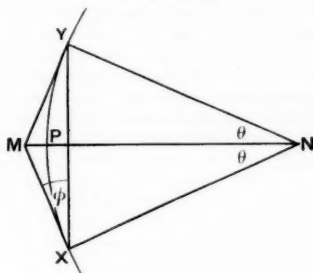


FIG. 1.

Let  $M$  be the point on the line  $NP$  not between  $N, P$  such that

$$|XM| + |MY| = |\widehat{XY}|.$$

Let the angle  $MYX$  be  $\phi$ . Then, by (4),

$$|MX| = \rho \sin \theta \sec \phi + o(\theta^3) \quad \dots\dots\dots(7)$$

so that, neglecting powers of  $\theta$  higher than the third,

$$\sin \theta / \theta = \cos \phi.$$

Hence

$$\phi = \theta / \sqrt{3} + o(\theta). \quad \dots\dots\dots(8)$$

Now, by (4), the angle  $PXY$  is  $\theta[\frac{1}{2} + o(1)]$ , and so

$$\text{angle } MXP = \theta \left[ \frac{1}{\sqrt{3}} - \frac{1}{2} + o(1) \right]. \quad \dots\dots\dots(9)$$

Hence, by (7) and (9), the area of the triangle  $MPX$  will be

$$a_2 = \theta^3 \left[ \frac{\rho^2}{2} \left( \frac{1}{\sqrt{3}} - \frac{1}{2} \right) + o(1) \right],$$

so that there exists a  $\theta_2 > 0$  such that, for  $0 < \theta < \theta_2$ ,

$$a_2 < \rho^2\theta^3/20. \quad \dots\dots\dots(10)$$

Now the angle between the tangent to  $A$  at  $X$  and the chord  $XY$  is  $\theta[1 + o(1)]$ , and so, by (8),  $XM$  will lie between the chord and the tangent for  $0 < \theta < \theta_3$ .

Suppose  $\theta = \theta_0 < \text{Min} [\theta_1, \theta_2, \theta_3]$ , and for this value of  $\theta$  the above construction is carried out. Let  $A'$  be the curve obtained from  $A$  by replacing the arc  $XY$  by the segments  $XM, MY$ . Then, by construction,  $A'$  will be in  $\Sigma$ . Further, by (6) and (10), the area enclosed by  $A'$  will be less than that enclosed by  $A$ . As this is a contradiction, the assertion (a) is proved. Thus each arc of  $A$  which is not a polygonal arc must be an arc of  $Q$ . We shall now see that

(b) If  $\pi$  is a polygonal arc forming part of  $A$ , every vertex of  $\pi$  lies on  $Q$ .

Suppose, if possible, that  $A, B, C$  are consecutive vertices of such an arc  $\pi$  and  $B$  does not lie on  $Q$ . Then there exists a point  $B'$  (lying on the ellipse through  $B$  with foci  $A, C$ ) such that

(i) the distance from  $B'$  to the line  $AC$  is less than the distance from  $B$  to the line  $AC$ ,

(ii) the curve  $A'$  obtained from  $A$  by replacing  $ABC$  by  $AB'C$  is convex and lies in  $H$ ,

(iii)  $|AB'| + |B'C| = |AB| + |BC|$ .

By (ii)  $A'$  belongs to  $\Sigma$ ; by (i) it encloses smaller area than  $A$ , and by (iii) it has the same length. This is impossible: hence the assertion (b) is proved.

Thus  $A$  consists of a finite or enumerable number of chords of  $Q$  together with, possibly, some arcs of  $Q$ . Neither the length of  $A$ , nor the area enclosed, is changed by assuming that it is made up of a single arc of  $Q$  subtending an angle  $\theta_0$  at the centre together with a number of chords of decreasing lengths subtending angles  $\theta_1, \theta_2, \dots$  at the centre of  $H$  ( $\theta_1 \geq \theta_2 \geq \dots$ ). Then, by definition of  $k(\theta)$  and  $a(\theta)$ ,

$$l = r \left[ 2\pi - \sum_{i=1}^{\infty} k(\theta_i) \right]$$

and

$$A = r^2 \left[ \pi - \sum_{i=1}^{\infty} a(\theta_i) \right]. \quad \dots\dots\dots (11)$$

We shall see that for the curve  $A$  the angles  $\theta_i$  are further restricted.

(c) With the above notation for  $A$ , there cannot exist integers  $i_2, i_3$  such that  $\theta_1 > \theta_{i_2} \geq \theta_{i_3}$  and  $1 < i_2 < i_3$ .

Suppose, if possible, there are integers  $i_2, i_3$  such that  $\theta_1 > \theta_{i_2} \geq \theta_{i_3}$ . Let  $\theta_{i_2}$  decrease by a small amount  $\delta\theta_3$ . Make increments  $\delta\theta_1$  in  $\theta_1$  and  $\delta\theta_2$  in  $\theta_{i_2}$  (actually  $\delta\theta_1$  will be negative) so that

$$\begin{aligned} \delta\theta_1 + \delta\theta_2 &= \delta\theta_3 \\ \delta k_1 + \delta k_2 &= \delta k_3, \quad \dots\dots\dots (12) \end{aligned}$$

where  $\delta k_1, \delta k_2, -\delta k_3$  are the increments in  $k(\theta_1), k(\theta_{i_2}), k(\theta_{i_3})$  corresponding to the increments  $\delta\theta_1, \delta\theta_2, -\delta\theta_3$ . If the corresponding increments in  $a(\theta_1), a(\theta_{i_2}), a(\theta_{i_3})$  are  $\delta a_1, \delta a_2, -\delta a_3$  we have, by the Lemma,

$$\delta a_3 > \delta k_3 \left( \frac{da}{dk} \right)_{\theta=\theta_{i_3}},$$

so that

$$\delta a_3 > \delta k_3 \left( \frac{da}{dk} \right)_{\theta=\theta_{i_1}},$$

since  $\theta_{i_2} \geq \theta_{i_3}$ , and

$$\delta a_1 + \delta a_2 < (\delta k_1 + \delta k_2) \left( \frac{da}{dk} \right)_{\theta=\theta_{i_2}} = \delta k_3 \left( \frac{da}{dk} \right)_{\theta=\theta_{i_1}}$$

by (12). Hence  $\delta a_1 + \delta a_2 < \delta a_3$ . Now, if  $A'$  is the curve obtained from  $A$  by replacing the sides subtending angles  $\theta_1, \theta_{i_2}, \theta_{i_3}$  by sides subtending angles  $(\theta_1 + \delta\theta_1), (\theta_{i_2} + \delta\theta_2), (\theta_{i_3} - \delta\theta_3)$   $A'$  will have length  $l$  and be of class  $\Sigma$ . However, by (11) it will enclose a smaller area than  $A$ . As this is impossible, the assertion (c) is proved.

(d)  $\theta_0 = 0$ : that is,  $A$  is a polygon.

Suppose, if possible, that  $\theta_0 > 0$ . Let  $AB$  be a small arc of  $A$  subtending an angle  $\phi$  at the centre, and  $BC$  the chord of  $A$  subtending  $\theta_1$  ( $\phi$  much smaller than  $\theta_1$ ). There is a point  $D$  on the arc  $BC$  of  $Q$  such that

$$|AD| + |DC| = |\widehat{AB}| + |BC|.$$

Then, if  $A'$  is obtained from  $A$  by replacing  $\widehat{AB}, BC$  by chords  $AD, DC$ ,  $A'$  belongs to  $\Sigma$ . The Lemma again shows that  $A'$  will enclose a smaller area than  $A$  which is a contradiction. Thus the assertion (d) is proved. Problem 1 is solved and  $A$  is determined completely. The result is stated in

**Theorem 1.** Suppose  $\Sigma$  is the class of convex curves of length  $l$  lying entirely in a circular disc of radius  $r$ , ( $4r < l < 2\pi r$ ). Among curves of  $\Sigma$ , the curve  $A$  which encloses the smallest area is the unique polygon of  $(n+1)$  sides ( $n \geq 2$ ) and length  $l$  such that

- (i) the vertices of  $A$  lie on the circumference of the disc,
- (ii)  $n$  of the sides of  $A$  have the same length  $a$ , while the remaining side has length  $b \leq a$ .

A result equivalent to Theorem 1 was proved by M. J. Favard in (1). There he finds which convex curves of given area and radius of circumscribed circle have the longest perimeter. The method of proof is, however, somewhat different.

In the same class  $\Sigma$  the curve  $\Gamma$  which encloses maximum area is trivially determined—it is simply a circle of radius  $l/2\pi$ . However, if the class is further restricted, we obtain the interesting result:

**Theorem 2.** Let  $\Pi$  be the class of convex curves of length  $l$  which are made up of arcs and chords of a circle of radius  $r$ , ( $0 < l \leq 4\pi r$ ). The curve  $\Gamma$  of  $\Pi$  which encloses maximum area is the boundary of the segment of a circle of radius  $r$  whose perimeter is  $l$ .

This theorem is a consequence of the Lemma. The method of proof is similar to that of the assertions (c), (d) leading to Theorem 1.

**2. Convex curves of given diameter.** Among convex curves of length  $l$  and diameter  $d$  ( $2d \leq l \leq \pi d$ ) it is known, see (4), that the curve which includes maximum area has the shape of a symmetrical lens of diameter  $d$ . However, the following analogous problem has not been solved.

**Problem 2.** What convex curve of given  $l$ ,  $d$  ( $2d \leq l \leq \pi d$ ) encloses the smallest area?

I cannot complete the solution to this problem, but below I give the result for certain values of  $l$ .

**Case (i)  $l = \pi d$ :** If a convex curve of diameter  $d$  has length  $\pi d$ , it must be a curve of constant width. It is known that the Reuleaux triangle\* is the curve of constant width which has smallest area. Thus in this case the answer to the problem is already known.

**Case (ii)  $2d \leq l \leq 3d$ :** The required curve is given by

**Theorem 3.** In the class of convex curves of length  $l$  and diameter not greater than  $d$  ( $2d \leq l \leq 3d$ ), the curve bounding minimum area is an isosceles triangle with sides  $d, d, l - 2d$ .

Let  $\Pi_n$  be the class of convex polygons of not more than  $n$  sides with given perimeter  $l$  and diameter not greater than  $d$ , ( $2d \leq l \leq 3d$ ). The theorem is proved by induction on  $n$ .

(i) The curve of  $\Pi_3$  enclosing minimum area is a triangle of sides  $d, d, l - 2d$ .

Suppose  $ABC$  is a triangle of  $\Pi_3$  with two of its sides of length less than  $d$ . If  $|AB| < d$ ,  $|AC| < d$ , then there is a triangle  $A'BC$  of smaller area and the same perimeter which belongs to  $\Pi_3$ . Hence, if the triangle  $ABC$  has the smallest possible area, not more than one of its sides can have length less than  $d$ , and the assertion is proved.

(ii) If the polygon of  $\Pi_n$  ( $n \geq 3$ ) of minimum area is an isosceles triangle, the same is true of  $\Pi_{n+1}$ .

\* Let  $A, B, C$  be three points such that  $AB = BC = CA$ . Join  $A$  to  $B$ ,  $B$  to  $C$ ,  $C$  to  $A$  by circular arcs with centres at  $C, A, B$ . The plane figure obtained is known as a Reuleaux triangle.

Suppose  $A, B, C$  are consecutive vertices of a polygon of  $\Pi_{n+1}$  such that  $B$  is at a distance less than  $d$  from all other vertices of the polygon. Then there exists a point  $B'$  whose distance from  $AC$  is less than that of  $B$  such that  $|AB'| + |B'C| = |AB| + |BC|$ . For a suitable  $B'$ , the polygon obtained by replacing  $ABC$  by  $AB'C$  will have a smaller area and will still be in  $\Pi_{n+1}$ . Hence every vertex of the polygon  $\Gamma$  of  $\Pi_{n+1}$  of minimum area must be at a distance  $d$  from at least one other vertex of  $\Gamma$ .

Suppose, if possible that  $\Gamma$  has five or more sides. Not more than two of these can be of length  $d$ . Therefore there must be at least one pair of consecutive sides of length less than  $d$ . Now a convex quadrilateral which has a pair of opposite sides each of length  $d$  must have diameter greater than  $d$  so that this case cannot arise. Hence, if  $\Gamma$  is a quadrilateral, there will again be two adjacent sides each of length less than  $d$ . The vertex common to these sides must be at a distance  $d$  from a vertex of  $\Gamma$  other than an adjacent one. Thus, if  $\Gamma$  has four or more sides there is at least one diagonal of  $\Gamma$  which has length  $d$ . This diagonal divides  $\Gamma$  into two polygonal arcs, and, considered with each of these in turn, gives rise to two convex polygons  $\Gamma_1, \Gamma_2$  each in  $\Pi_n$ . Let  $ABC$  be a triangle with the same perimeter as  $\Gamma_1$  and  $|AB| = |AC| = d$ . Let  $ADC$  be a triangle with the same perimeter as  $\Gamma_2$ ,  $|AD| = |AC| = d$ , and  $D, B$  on opposite sides of line  $AC$ . Then the quadrilateral  $ABCD$  has the same perimeter as  $\Gamma$ . As  $l \leq 3d$ , it follows that  $|BD| < d$  and so the quadrilateral belongs to  $\Pi_{n+1}$ . By hypothesis on  $\Pi_n$ , unless each of  $\Gamma_1, \Gamma_2$  is already an isosceles triangle, the quadrilateral  $ABCD$  will have a smaller area than  $\Gamma$  which is impossible. Thus, if  $\Gamma$  has four or more sides, it must be a quadrilateral which can be denoted  $ABCD$  with  $|AB| = |AC| = |AD| = d$ . But the area of such a quadrilateral is the sum of areas of isosceles triangles with bases  $BC, CD$  and altitudes greater than that of the isosceles triangle with base of length  $|BC| + |CD|$ . Thus it is greater than the area of a triangle of sides  $d, d, l - 2d$ . Since this is impossible,  $\Gamma$  can only be a triangle of sides  $d, d, l - 2d$ .

Since the figure of  $\Pi_3$  of minimum area is, by (i), an isosceles triangle, the same must be true of  $\Pi_n$  for  $n \geq 4$ . Now a convex curve of length  $l$  and width not greater than  $d$  is a limit of polygons of  $\Pi_n$  as  $n \rightarrow \infty$ . Thus the theorem is proved.

I do not think a direct proof of the result contained in theorem 3 has been given previously. See page 82 of (2), or (3) for previous results from which the theorem may be deduced.

Case (iii)  $3d < l < \pi d$ : For this range I have been unable to find the convex curve of given  $l, d$  which encloses the smallest area. However, the following result is of interest.

**Theorem 4.** *In the class  $\Pi_4$  of convex quadrilaterals of perimeter  $l$  and diameter not exceeding  $d$  ( $3d < l \leq \pi d$ ), the quadrilateral  $\Gamma$  which encloses minimum area is inscribed in a Reuleaux Triangle of width  $d$ ; three of the vertices of  $\Gamma$  coincide with the vertices of the Reuleaux Triangle. ( $\tau$  is the largest perimeter of a convex quadrilateral of width  $l$ .)*

Let  $ABCD$  be the quadrilateral  $\Gamma$  which encloses the smallest area; and suppose, if possible, that no side of  $\Gamma$  has length  $d$ . Then, as in the proof of Theorem 3, each of the diagonals  $AC, BD$  has length  $d$ . Let  $BN, DM$  be the perpendiculars from  $B, D$  to  $AC$ . There is no loss in generality in assuming that  $|AM| \geq |AN| \geq \frac{1}{2}d$ . Since  $|AM| = |AN| = \frac{1}{2}d$  would imply that  $l < 3d$  and this is irrelevant to the range under consideration, we may suppose that either  $|AM| \geq |AN| > \frac{1}{2}d$  or  $|AM| > |AN| = \frac{1}{2}d$ . In either case  $B$  can be replaced by  $B'$  nearer  $AC$ , and also nearer  $D$ , than  $B$  while  $AB'CD$  is in  $\Pi_4$ . This new quadrilateral will have smaller area than  $\Gamma$  which is a contradiction. Hence  $\Gamma$  must have at least one side of length  $d$ .

Suppose  $AD$  has length  $d$ . Let  $AX, DX$  be arcs of radius  $d$  and centres  $D, A$  respectively with  $X$  on the same side of  $AD$  as  $B, C$ . Suppose, if possible that neither of  $B, C$  coincide with  $X$ . Then each must be on one of the arcs  $AX, DX$  and two cases are possible.

(a) Firstly, both  $B, C$  may be on the same arc, say on  $AX$  (see Fig. (ii)). There is no loss in generality in assuming that  $|AB| \leq |BC|$ . Let  $P$  be the point on  $AB$  such that  $|PB| + |BC| = |PX|$ . Then the length of the perpendicular from  $D$  to  $PX$  is less than the perpendicular to  $BC$  which, in turn, is not greater than the perpendicular to  $AB$ . Thus the area of the triangle  $PXD$  is less than the sum of the areas of  $PBD, BCD$ , and this implies that the quadrilateral  $APXD$  has smaller area than  $ABCD$  which is impossible.

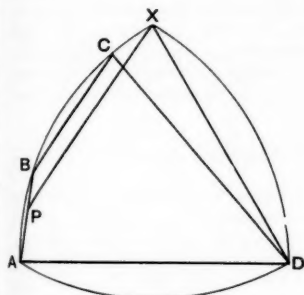


FIG. 2.

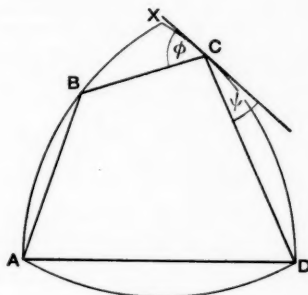


FIG. 3.

(b) Secondly, we may have  $B$  on the arc  $AC, C$  on the arc  $DX$ . We lose no generality in assuming that  $|BX| \geq |CX|$  (see Fig. (iii)). As  $l > 3d$  it follows that  $|BC| < |CD|$ . Let  $BC$  make an angle  $\phi$  with the tangent to  $DX$  at  $C$ . Then, since the angle between the tangents to  $AX, DX$  at  $X$  is  $\pi/3$ ,  $\phi > \pi/6$ . Also, since the angle subtended by  $CD$  at the centre  $A$  is less than  $\pi/3$ ,  $\psi$ , the angle between  $CD$  and  $t$ , satisfies  $\psi < \pi/6$ . Thus  $\phi > \psi$ , and the ellipse through  $C$  with foci  $B, D$  crosses the arc  $DX$  at  $C$  from the outside of the Reuleaux triangle to the inside if we move in the direction which makes the distance to  $B$  decrease. Hence there will be a point  $C'$  on this ellipse inside the Reuleaux triangle and nearer to  $BD$  than  $C$ . The quadrilateral  $ABC'D$  would belong to  $\Pi_1$  and have smaller area than  $F$  which is impossible.

It follows that one of  $B, C$  must coincide with  $X$ , and the Theorem is proved.

**Corollary.** All convex quadrilaterals of diameter  $d$  have perimeter not greater than  $\pi d$  where  $\tau = 2 + 4 \sin(\pi/12)$ .

**Proof:** Among all quadrilaterals of diameter  $d$  and perimeter  $l$  we have proved that the one of smallest area can be denoted  $ABCD$  with

$$|AB| = |AC| = |AD| = |BD| = d.$$

The perimeter of a quadrilateral of this type is a maximum when

$$|BC| = |CD| :$$

in this case  $l$  has the value  $\pi d$  with  $\tau = 2 + 4 \sin(\pi/4)$ . This completes the proof of the Corollary and also shows that the upper bound is attained.

We shall see in the next section that even for the range  $(3d < l \leq \pi d)$  the convex curve of given  $l, d$  which encloses smallest area is never a quadrilateral so that the minimum found above does not apply to the wider class of convex

curves of given  $l, d$ . However the result of Theorem 4 suggests that the convex curve enclosing smallest area is likely to be inscribed in a Reuleaux triangle. Though I can prove that the curve of  $\Pi_3$  which encloses minimum area has this property, I cannot either prove or disprove the supposition for  $\Pi_n$  ( $n \geq 6$ ). If it is true that the curve of given  $l, d$  which encloses minimum area is inscribed in a Reuleaux triangle, then the results of the next section will give a complete solution to Problem 2.

3. *Convex curves lying in a Reuleaux triangle*: I now consider the following:

**Problem 3.** Let  $\Sigma$  be the class of convex curves of length  $l$  lying in a Reuleaux triangle  $\Delta$  of width  $d$ , ( $0 < l < \pi d$ ). What curves of  $\Sigma$  enclose the smallest area?

Case (i)  $0 < l \leq 2d$ : As in § 1 this is trivial, the required curve being a segment of length  $\frac{1}{2}l$ .

Case (ii)  $2d < l \leq 3d$ : Any convex curve lying in  $\Delta$  has width at most  $d$ . The curve of length  $l$  and width at most  $d$  which bounds the smallest area is, by Theorem 3, a triangle with sides of length  $d, d, l - 2d$ . But such a triangle is in the smaller class  $\Sigma$  and so must give the minimum area for that class also.

Case (iii)  $3d < l < \pi d$ : Let  $A$  be the curve of  $\Sigma$  which encloses minimum area. Similar methods to those used in § 1 for curves lying in a disc will show that

(a)  $A$  is a polygon,

(b) the vertices of  $A$  lie on  $\Pi$ , the boundary of  $\Delta$ .

If we let  $\Pi_1, \Pi_2, \Pi_3$  be the three circular arcs of radius  $d$  making up  $\Pi$ , then similar methods will also show that,

(c) if  $P_n, P_{n+1}, P_{n+2}, \dots, P_{n+p}$  are  $(p+1)$  consecutive vertices of  $A$  lying on  $\Pi_i$ , ( $p \geq 2$ ), ( $i$  fixed = 1, 2, or 3); then  $p-1$  of the chords  $P_j, P_{j+1}$  ( $n \leq j \leq n+p-1$ ) have the same length  $q$  and the length of the remaining one is not greater than  $q$ .

We need to examine how, in such a configuration, the area increases relative to the length. Suppose the conditions of (c) are satisfied and  $P_n, \dots, P_{n+p}$  are points on the arc of a circle of radius  $d$  and centre  $O$ . Let the angle  $P_n O P_{n+p}$  have the fixed value  $\alpha \leq \pi/3$ , and suppose  $P_n P_{n+1}$  subtends an angle  $\theta$  at  $O$  while each of  $P_j P_{j+1}$  ( $n+1 \leq j \leq n+p-1$ ) subtend the same angle  $(\alpha - \theta)/p$  at  $O$ . Then the length of the polygonal arc  $P_n P_{n+1} \dots P_{n+p}$  is given by

$$l = 2d \left[ \sin \frac{\theta}{2} + p \sin \frac{1}{2p} (\alpha - \theta) \right],$$

so that 
$$\frac{dl}{d\theta} = d \left[ \cos \frac{\theta}{2} - \cos \frac{1}{2p} (\alpha - \theta) \right]. \dots\dots\dots (14)$$

Also the area of the polygon bounded by this polygonal arc and the radii  $P_n O, P_{n+p} O$  is given by

$$A = \frac{1}{2} d^2 \left[ \sin \theta + p \sin \frac{1}{p} (\alpha - \theta) \right],$$

so that 
$$\frac{dA}{d\theta} = \frac{1}{2} d^2 \left[ \cos \theta + \cos \frac{1}{p} (\alpha - \theta) \right]. \dots\dots\dots (15)$$

Thus, as  $\theta$  increases, each of  $l, A$  increase and, by (14) and (15),

$$\frac{dA}{dl} = d \left[ \cos \frac{\theta}{2} + \cos \frac{1}{2p} (\alpha - \theta) \right]. \dots\dots\dots (16)$$

Further, by (c) for the relevant figures,  $\theta \leq (\alpha - \theta)/p$  and so, since  $p \geq 2$ ,  $\theta \leq \pi/6$ . Hence, by (16),  $dA/dl \geq d[\cos \pi/12 + \cos \pi/6]$  and so

$$dA/dl > 7d/4 \dots\dots\dots (17)$$

for small allowable changes in the configurations of (c). We can now prove that, if  $U, V, W$  are vertices of  $\Delta$ ,

(d) each of  $U, V, W$  is a vertex of  $A$ .

Suppose, if possible, that the vertex  $U$  of  $\Delta$  is not on  $A$ . Let  $BC$  be a side of  $A$  such that  $B$  is on the arc  $UW$ , and  $C$  is on the arc  $UV$ . There is no loss in generality in assuming that  $|BU| \geq |CU|$ . We consider two distinct possibilities for the curve  $A$ :

(i) Firstly, suppose there is an  $i$  ( $i=1, 2$  or  $3$ ) such that three or more vertices of  $A$  lie on  $\Pi_i$ . Let  $BC$  make an angle  $\phi$  with the tangent at  $C$  to  $UV$

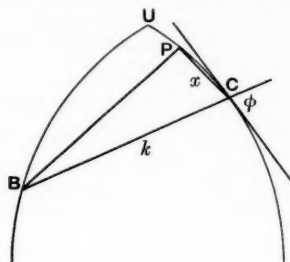


FIG. 4.

(see Fig. (iv)). Let  $P$  be a point on  $UV$  at a small distance  $x$  from  $C$ . Then the area of the triangle  $BPC$  is given by

$$\delta A = \frac{1}{2} x k \sin \phi + o(x), \dots\dots\dots(18)$$

where  $k$  is the length of  $BC$ . The difference  $|BP| + |PC| - |BC|$  is given by

$$\delta l = x(1 - \cos \phi) + o(x). \dots\dots\dots(19)$$

Hence, from (18) and (19),

$$\delta A / \delta l = \frac{1}{2} k \cot \frac{1}{2} \phi + o(x). \dots\dots\dots(20)$$

Now, it is clear that  $0 \leq k \leq d$ ,  $\phi > \pi/6$ . Thus, if  $k \leq \frac{1}{2}d$ , it follows from (20) that

$$\delta A / \delta l < \frac{1}{4} d \cot \pi/12 + o(x).$$

On the other hand, if  $k > \frac{1}{2}d$ , the geometry of the figure shows that  $\phi > \pi/5$ , and so

$$\delta A / \delta l < \frac{1}{2} d \cot (\pi/10) + o(x).$$

Hence, in any case, for sufficiently small  $x$ ,

$$\delta A / \delta l < 7d/4. \dots\dots\dots(21)$$

Suppose now that we obtain the polygon  $A'$  from  $A$  by the following changes :  
(a) Carry out the above construction for small  $x$  and replace  $BC$  by the segments  $BP, PC$ . This increases the perimeter by  $\delta l$  and the area by  $\delta A$ .

(b) By hypothesis there is at least one polygonal arc  $\pi$  of  $A$  inscribed in a side  $\Pi_i$  such that there are two or more segments in  $\pi$ . In the above notation replace  $\pi$  by  $\pi'$  obtained by reducing  $\theta$  so that the length is reduced by  $\delta l$ . If this reduces the area of  $A$  by  $\delta A'$  it follows from (17) and (19) that

$$\delta A < (7d/4)\delta l \text{ and } \delta A' > (7d/4)\delta l$$

so that  $\delta A' > \delta A$ . Thus there exists a new polygon  $A'$  with the same perimeter as  $A$  but enclosing a smaller area : this is impossible.

(ii) Secondly, suppose that on each of  $\Pi_i$  ( $1 \leq i \leq 3$ ) there are at most two

vertices of  $\Delta$ . Let  $A, D, E$  be such that  $A, B, C, D, E$  are consecutive vertices of  $\Delta$ . Let  $AB, BC$  make angles  $\phi_1, \phi_2$  with the tangent at  $B$ , and let  $BC, CD$  make angles  $\psi_1, \psi_2$  with the tangent at  $C$ . We first consider the polygonal arc  $BCD$ .

Suppose if possible, that  $|CD| > |BC|$ . Then, by considering the ellipse through  $C$  with foci  $B, D$  it is clear that  $\psi_2 > \psi_1$ . Since  $\psi_1 > \pi/6, \psi_2 > \pi/6$  and  $D$  is on the arc  $VW$ . Suppose first that  $|CV| \leq |CU|$ ; and let  $P$  be on the arc  $VW$  with  $|CP| = |BC|$ . Then  $|WP| \leq |WB|$ , and since  $|CD| > |BC|$ ,  $|WD| < |WB|$ . If, on the other hand  $|CV| > |CU|$ , let  $P$  be on the arc  $VW$  such that the angles  $BCW, WCP$  are equal. Then we again have

$$|WP| \leq |WB|$$

and, since  $\psi_2 > \psi_1$ ,  $|WD| < |WB|$ . Thus we see that  $|CD| > |BC|$  implies that  $D$  is on the arc  $VW$  and  $|WD| < |WB|$ . This means that there would be a triangle  $BC'D$  similar to  $DCB$  with  $C'$  inside  $\Delta$ . But this is impossible. Therefore  $|CD| \leq |BC|$ . As before, this implies that  $\psi_2 \leq \psi_1$ .  $|UD| > |UB|$  is impossible as this would again imply the existence of a triangle  $BC'D$  similar to  $DCB$  with  $C'$  inside  $\Delta$ . Hence  $|UD| \leq |UB|$ , and this incidentally proves that  $|UB| = |UC|$  is impossible.

Now consider the polygonal arc  $ABC$ . Suppose first that  $|BW| \leq |BU|$ . Similar arguments to those just used will show that  $A$  must be on the arc  $VW$  with  $|WA| \geq |UC|$ ,  $|AB| \leq |BC|$ ,  $\eta_1 \leq \phi_2$ . If, on the other hand

$$|BW| > |BU|,$$

then  $A$  may be on either of the arcs  $VW, WU$  provided  $|AB| > |BC|$ ,  $\phi_1 > \phi_2$ , and, if  $A$  is on  $VW$ ,  $|WA| < |UC|$ .

Now, since  $C, D$  are on the arc  $UV$  and there are not more than three vertices on any one arc,  $E$  can only be on the arc  $VW$ . There are two possible cases represented by Figures (v) and (vi).

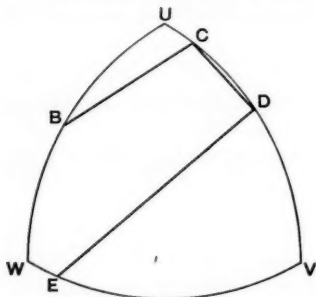


FIG. 5.

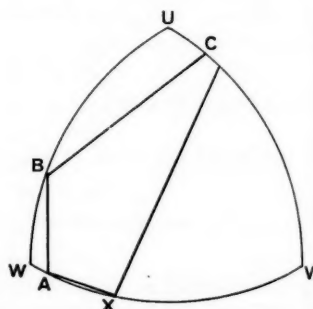


FIG. 6.

( $\alpha$ ) Suppose  $|BU| \leq |BW|$ . Then  $|DU| \leq |DV|$  and, in consequence, by applying the same arguments to  $DE$  as have been applied to  $BC$ ,  $|DE| > |CD|$  and  $|VE| \geq |VC|$ . Hence  $|WE| \leq |UC|$ , and the next vertex after  $E$  must be  $B$ . If we now apply the same arguments to  $BE$ , this is seen to be impossible since  $|WE| > |WB|$ .

( $\beta$ ) Suppose now that  $|BU| > |BW|$ . Since  $|AB| > |BC|$  it follows that  $|AW| > |BW|$ . Also, if  $X$  is the vertex of  $\Delta$  before  $A$ , then  $X$  is on the arc  $VW$  with  $|WX| \leq |WB|$  and therefore  $|XW| < |XV|$ . As  $D, X$ , can only

be consecutive vertices,  $|DV| > |XV|$ : this implies that  $|DU| < |DV|$  which again gives a contradiction when the polygonal arc  $CDX$  is considered.

In this way, it is seen that  $A$  cannot be such that there are not more than two vertices on each of the arcs of  $\Delta$ . Hence there are no polygons  $A$  which enclose minimum area and do not contain all the vertices of  $\Delta$ , and the assertion (d) is proved.

Thus we see that we can think of  $A$  as the sum of polygonal arcs  $A_1, A_2, A_3$  inscribed in  $\Pi_1, \Pi_2, \Pi_3$  with end-points at the vertices of  $\Delta$ . From the assertion (c), it follows that  $A_i$  ( $i=1, 2, 3$ ) is made up of  $p_i$  equal chords together with a further one of the same, or smaller, length. Suppose  $A_i$  has length  $l_i$  ( $i=1, 2, 3$ ), and  $A_i$  is the area between  $A_i$  and the single chord with the same end-points. From (16) we have, since  $\alpha = \pi/3$ ,

$$\frac{dA_i}{dl_i} = d \left[ \cos \frac{\theta_i}{2} + \cos \frac{1}{2p_i} \left( \frac{\pi}{3} - \theta \right) \right], \dots\dots\dots(22)$$

where  $\theta_i$  is the angle subtended by the smaller chord of  $A_i$ , and the other chords of  $A_i$  subtend the same angle  $(1/p_i)(\pi/3 - \theta_i)$ . A careful examination of  $dA_i/dl_i$  shows that its graph as a function of  $l_i$  looks something like Fig. (vii)

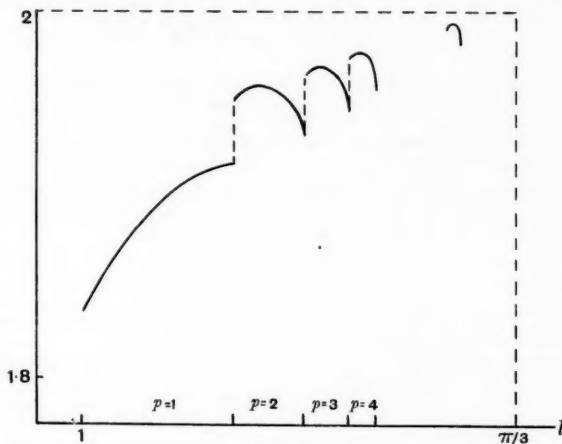


FIG. 7.

There are discontinuities for the values of  $l_i$  at which  $p_i$  increases by 1. Careful analysis of the properties of (22) and its differential coefficients with respect to  $l_i$  can be applied by the methods already used to prove the assertions

(e) If  $i_1, i_2$  ( $1 \leq i_1 < i_2 \leq 3$ ) are such that  $\theta_{i_1} > 0$ , then  $p_{i_1} = p_{i_2}$  and  $\theta_{i_1} = \theta_{i_2}$ .

(f) If  $i_1, i_2$  ( $1 \leq i_1 < i_2 \leq 3$ ) are such that  $p_{i_1} < p_{i_2}$ , then  $\theta_{i_1} = 0$  or  $\theta_{i_2} = 0$  and  $p_{i_1} + 1 = p_{i_2}$ .

For a given value of  $l/d$ , this means that there are a finite number of possibilities (not more than 6) for the values of  $l_1, l_2, l_3$ . By actual evaluation the area of the curves  $A$  which would result from these possibilities can be found, and so the configuration which makes the area smallest can be chosen. To see that both of the possibilities considered in the assertions (e) and (f) are possible consider the following special values.

(i) Let  $d = 1$ ,  $l = 3 \cdot 10 < 12 \sin(\pi/12)$ . Then, if  $l_1 \leq l_2 \leq l_3$ ,  $p_1 = 1$  and so  $p_2 \leq 2$ ,  $p_3 \leq 2$ . However, in the range  $1 \leq l_i \leq 4 \sin(\pi/12)$ ,  $dA_i/dl_i$  is strictly increasing in  $l_i$ , and so  $p_1 = p_2 = p_3$ ,  $\theta_1 = \theta_2 = \theta_3$ . Incidentally, this shows that, for

$$3d < l \leq \pi d$$

the convex curve which encloses minimum area is not a quadrilateral (see Theorem 4), since there is a hexagon of smaller area inscribed in a Reuleaux triangle.

(ii)  $d = 1$ ,  $l = 3 \cdot 111$ . Working out the possibilities in this case shows that  $A$  has 7 sides with, say,  $A_1$ ,  $A_2$ , each having 2 segments and  $A_3$  having 3 segments (one of which is shorter than the other two). If, on the other hand,  $l = 3 \cdot 113$ , the polygon  $A$  will have 8 sides with, say,  $A_1$  having 3 equal segments and  $A_2$ ,  $A_3$  having one segment shorter than the other two.

The results are collected in

**Theorem 5.** Suppose  $\Sigma$  is the class of convex curves of length  $l$  lying in a Reuleaux triangle of width  $d$  ( $3d \leq l \leq \pi d$ ) and  $A$  is a member of  $\Sigma$  which encloses the smallest area: then  $A$  is a polygon made up of three polygonal arcs  $A_1$ ,  $A_2$ ,  $A_3$  such that

- (i) every vertex of  $A_i$  ( $i = 1, 2, 3$ ) is on the Reuleaux triangle and the end-points of  $A_i$  are two vertices of the Reuleaux triangle,
- (ii)  $A_i$  consists of  $p_i$  equal segments together with, possibly, a further smaller segment,
- (iii) if, in two of the polygonal arcs making up  $A$  there are different numbers of segments, this difference cannot be more than 1, and at least one of the arcs considered must have all its segments equal,
- (iv) if there are two polygonal arcs which each have a segment unequal to the others, then these arcs must have the same length.

It is interesting to see how  $A$  changes shape for  $d = 1$  and  $l$  increasing in the range  $(3, \pi)$ . The number of sides in the polygon goes through the values 3, 6, 9, 7, 9, 8, 9, 12, 10, 12, 11, 12, 15, ... in that order. S. J. T.

## REFERENCES.

1. M. J. Favard, "Problèmes d'extremums relatifs aux courbes convexes", *Annales de l'école normale supérieure*, 46 (1929), 344-369.
2. Bonneson and Fenchel, "Theorie der konvexen korper", *Ergebnisse der Mathematik*, Vol. 3, Berlin (1934).
3. T. Kubota, "Einige Ungleichheitsbeziehungen uber eiliniien und eiflachen", *Tohoku Imp. Univ. Sc. Rep.*, 1, No. 12 (1923-4), 45-65.
4. A. S. Besicovitch, "Some variants of the classical isoperimetric problem". *Quart. J. Math.*, Oxford Ser. (2), 3, (1952), 42-49

1750. The recent analysis undertaken by Dr. Enid Charles shows that whatever changes in fertility and mortality may conceivably conspire to avert a rapid decline of net population from 1945 onwards, nothing can now forestall a rapid and spectacular depletion of the school age groups during the next two decades. Therefore the choice lies between a period of acute unemployment for teachers or a drastic reform of educational routine. No teacher should teach for more than ten hours a week. By 1950 an enormous reduction of working hours can be achieved without any increase in the cost of education.—L. Hogben, *Math. Gazette*, 22 (1938), 122.

## INVOLUTION.

BY E. H. NEVILLE.

It is a sound principle in teaching that we should begin by saying what a thing is, rather than what it is not, but the reader of the *Gazette* is not coming across the word involution for the first time, and the best service I can render him is to insist at the outset that involution is not a form of homography. Before I can enlarge on this warning, I must add another. In elementary work we are in no danger of confusing the relation of collinearity, a relation between three or more points, with a class of collinear points, for which we have the familiar name of line. But the one word involution has been made to do duty both for a relation and for a class, and although when we understand the subject we can always interpret the word from the context, the teacher will find it worth while to sacrifice brevity for a time and to talk of the involution-relation and of an involution-class.

A homography is a relation between members of one family and members of another; it is possible, but not relevant, for the families to coincide, in which case the homography is a linear transformation within the family, but the terms between which the homography holds are always single elements. The involution-relation, on the other hand, is fundamentally a relation between pairs of elements, and an involution-class is a class of pairs.

If an individual member  $U$  of a family of elements is determined by the value of a parameter  $x$ , a pair of members  $U_1, U_2$  can be determined by the quadratic equation  $a_w x^2 + 2b_w x + c_w = 0$  whose roots are the values  $u_1, u_2$  of  $x$  which correspond to  $U_1, U_2$ . The set of coefficients  $a_w, b_w, c_w$ , or rather the set of ratios  $a_w : b_w : c_w$ , determines the pair of parameters without specifying the members separately, and so relieves us of the need of reiterating that it is between pairs that the relation now to be studied subsists. We can return to the two components if we wish, or rather to their sum and product, for we have

$$a_w x^2 + 2b_w x + c_w = a_w (x - u_1)(x - u_2),$$

identically. In the analytical geometry of conics, it is always by means of equations which we do not solve that we discuss points of intersection, pairs of tangents, and the like.

The pair of elements  $W_1, W_2$ , or briefly the pair  $W$ , given by the set of coefficients  $a_w, b_w, c_w$  is said to be in involution-relation with the pairs  $U$  and  $V$  given by the sets  $a_u, b_u, c_u$  and  $a_v, b_v, c_v$  if the quadratic function  $a_w x^2 + 2b_w x + c_w$  is a linear combination of the two quadratic functions

$$a_u x^2 + 2b_u x + c_u, \quad a_v x^2 + 2b_v x + c_v,$$

that is, if there are constants  $\mu, \nu$  such that

$$a_w x^2 + 2b_w x + c_w \equiv \mu(a_u x^2 + 2b_u x + c_u) + \nu(a_v x^2 + 2b_v x + c_v).$$

The involution-relation is symmetrical in the three pairs of elements concerned, and it is expressed analytically in terms of the coefficients by the vanishing of the determinant

$$\begin{vmatrix} a_u & b_u & c_u \\ a_v & b_v & c_v \\ a_w & b_w & c_w \end{vmatrix}.$$

In terms of sums and products of parameters, the involution-relation is

$$\begin{vmatrix} u_1 u_2 & u_1 + u_2 & 1 \\ v_1 v_2 & v_1 + v_2 & 1 \\ w_1 w_2 & w_1 + w_2 & 1 \end{vmatrix} = 0. \dots\dots\dots (1)$$

An alternative form of the relation between the six separated parameters lends itself better to geometrical arguments. Since  $a_w w_1^2 + 2b_w w_1 + c_w = 0$  and  $a_w w_2^2 + 2b_w w_2 + c_w = 0$ , we have

$$\mu a_u (w_1 - u_1)(w_1 - u_2) + \nu a_v (w_1 - v_1)(w_1 - v_2) = 0,$$

$$\mu a_u (w_2 - u_1)(w_2 - u_2) + \nu a_v (w_2 - v_1)(w_2 - v_2) = 0.$$

and the result of eliminating  $\mu a_u : \nu a_v$  between these equations can be written

$$\frac{(w_1 - u_1)/(w_1 - v_1)}{(w_2 - u_1)/(w_2 - v_1)} = \frac{(w_2 - u_2)/(w_2 - v_2)}{(w_1 - u_2)/(w_1 - v_2)} \dots \dots \dots (2)$$

This equality of cross-ratios can be written in terms of the geometrical elements themselves :

$$(U_1 V_1, W_1 W_2) = (U_2 V_2, W_2 W_1) \dots \dots \dots (3)$$

The conditions (1), (2) can not be identical, for (2) is not symmetrical in the three pairs of variables. To find the relation between them, we have

$$(w_1 - u_1)(w_1 - u_2) = w_1 \{ (w_1 + w_2) - (u_1 + u_2) \} - (w_1 w_2 - u_1 u_2),$$

identically, whence

$$\begin{vmatrix} (w_1 - u_1)(w_1 - u_2) & (w_2 - u_1)(w_2 - u_2) \\ (w_1 - v_1)(w_1 - v_2) & (w_2 - v_1)(w_2 - v_2) \end{vmatrix} \\ = \begin{vmatrix} w_1 & -1 \\ w_2 & -1 \end{vmatrix} \begin{vmatrix} (w_1 + w_2) - (u_1 + u_2) & w_1 w_2 - u_1 u_2 \\ (w_1 + w_2) - (v_1 + v_2) & w_1 w_2 - v_1 v_2 \end{vmatrix} \\ = (w_1 - w_2) \begin{vmatrix} u_1 u_2 & u_1 + u_2 & 1 \\ v_1 v_2 & v_1 + v_2 & 1 \\ w_1 w_2 & w_1 + w_2 & 1 \end{vmatrix}.$$

Thus it is only when the cross-ratio equality is trivial that it fails to imply the involution-relation.

The pairs in involution with two given pairs  $U, V$  form a class of pairs, the involution-class based on  $U$  and  $V$ . If the two quadratic functions  $a_w x^2 + 2b_w x + c_w$ ,  $a_v x^2 + 2b_v x + c_v$  are both linear combinations of  $a_u x^2 + 2b_u x + c_u$  and  $a_v x^2 + 2b_v x + c_v$ , then every quadratic function which is a linear combination of the first two is necessarily a linear combination of the last two. That is, if the pairs  $W, T$  both belong to the involution-class based on  $U$  and  $V$ , then every member of the involution-class based on  $W$  and  $T$  belongs to the involution-class based on  $U$  and  $V$ . But since, if  $W$  and  $T$  are distinct, the quadratic functions  $a_w x^2 + 2b_w x + c_w$  and  $a_v x^2 + 2b_v x + c_v$  can be expressed as linear combinations of  $a_u x^2 + 2b_u x + c_u$  and  $a_v x^2 + 2b_v x + c_v$ , the pairs  $U, V$  belong to the involution-class based on  $W$  and  $T$ , and therefore every member of the involution-class based on  $U$  and  $V$  belongs to the involution-class based on  $W$  and  $T$ . Thus the two classes are identical ; the involution-class is the same class, from whatever two or its members it is constructed.

If  $U_1$  coincides with  $U_1$  and  $V_2$  with  $V_1$ , the relation (3) becomes

$$(U_1 V_1, W_1 W_2) = (U_1 V_1, W_2 W_1),$$

and implies, since one of these cross-ratios is identically the reciprocal of the other, that the value of  $(U_1 V_1, W_1 W_2)$  is either 1 or -1, that is, that either  $W_1$  coincides with  $W_1$  or the pair  $(W_1, W_2)$  harmonises with the pair  $(U_1, V_1)$ . If  $u_2 = u_1, v_2 = v_1, w_2 = w_1$ , then (1) becomes

$$\begin{vmatrix} u_1^2 & u_1 & 1 \\ v_1^2 & v_1 & 1 \\ w_1^2 & w_1 & 1 \end{vmatrix} = 0$$

that is  $(u_1 - v_1)(v_1 - w_1)(w_1 - u_1) = 0$ , and this means that we are not in fact dealing with three distinct pairs. Hence  $W_2$  does not coincide with  $W_1$ , and the second alternative holds. That is, an involution-class can not include more than two pairs of coincident members, or double elements as they are called, and if it does include two such pairs  $(X, X)$  and  $(Y, Y)$ , the class consists of the pairs of elements harmonic with the pair  $(X, Y)$ . It follows that if the elements are points on a line, and if  $O$  is the mid-point of  $XY$ , the product  $OW_1 \cdot OW_2$  has the constant value  $OX^2$  for every point-pair  $(W_1, W_2)$  belonging to the involution-class.

To discover whether double elements are to be expected in an involution-class, we return to (1), which we can take in the form

$$\begin{vmatrix} c_u & -2b_u & a_u \\ c_v & -2b_v & a_v \\ w_1w_2 & w_1 + w_2 & 1 \end{vmatrix} = 0. \quad (4)$$

This is an equation of the form

$$Aw_1w_2 + B(w_1 + w_2) + C = 0, \quad (5)$$

and since the determinantal equation is unaltered if the first two rows are replaced by linear combinations of these rows, the ratios  $A : B : C$  are unaltered if  $U, V$  are replaced by any two members of the involution-class constructed on them; this is inevitable, for  $W$  could not be a variable member of the class if the constituent elements  $w_1, w_2$  were connected by more than one equation. The constituents have the same value  $w'$ , and  $W$  is a double element  $(W', W')$ , if  $w'$  satisfies the quadratic equation

$$Aw'^2 + 2Bw' + C = 0. \quad (6)$$

Hence in complex geometry an involution-class in general includes two double elements  $(X, X), (Y, Y)$  and consists of the pairs which harmonise with the pair  $(X, Y)$ . In real geometry, involution-classes are of two kinds, those which have double elements and those which have not.

If only  $A \neq 0$ , then whether (6) has real roots or not, (5) can be written in the form

$$(Aw_1 + B)(Aw_2 + B) = B^2 - AC,$$

that is,

$$A^2(w_1 - w')(w_2 - w') = B^2 - AC, \quad (7)$$

where  $w' = -B/A$ . In particular, if the elements are points on a line, distance along the line is a possible parameter and the relation between the two components of a point-pair belonging to the involution-class is

$$OW_1 \cdot OW_2 = k.$$

The pupil must be warned that while double values in any parameter correspond to double elements of the involution-class, an involution-class as such has no intrinsic centre. In (7), if  $w_1 = w'$ , then  $w_2$  can not have any finite value; that is, the centre-value of the parameter is paired with the infinite value, and since the infinite value of one parameter does not necessarily correspond to the infinite value of another, neither does the centre-value of one to the centre-value of another. For points on a line, the point at infinity often has a special part to play, and then the companion of this point is equally important as a centre in any involution-class of point-pairs on the line. But an involution-class of line-pairs with a fixed vertex, or of point-pairs on a conic, has no centre.

A different line of argument illustrates the use of homogeneous coordinates. If the variable element is associated not with the value of one parameter  $x$

but with the ratio of the values of two parameters  $x, y$ , the pairs of elements  $U, V$  are determined by homogeneous equations

$$a_u x^2 + 2b_u xy + c_u y^2 = 0, \quad a_v x^2 + 2b_v xy + c_v y^2 = 0,$$

and the combination

$$a_v(a_u x^2 + 2b_u xy + c_u y^2) - a_u(a_v x^2 + 2b_v xy + c_v y^2) = 0$$

is satisfied by  $y = 0$  and by only one finite value of  $x/y$ ; this value is the centre-value associated with  $x/y$  as a non-homogeneous parameter.

The equation (5) invites comparison with the equation

$$A\lambda\mu + B(\lambda + \mu) + C = 0 \quad \dots\dots\dots(8)$$

which defines a symmetrical homography between the parameters  $\lambda, \mu$  of members of distinct families. The comparison provides an alternative proof of the cross-ratio property; we know that if  $\lambda_1, \lambda_2, \lambda_3, \lambda_4$  are any four values of  $\lambda$ , and  $\mu_1, \mu_2, \mu_3, \mu_4$  the values of  $\mu$  connected with them by (8), then

$$(\lambda_1\lambda_2, \lambda_3\lambda_4) = (\mu_1\mu_2, \mu_3\mu_4)$$

and if we take  $\lambda_1 = u_1, \lambda_2 = v_1, \lambda_3 = w_1, \lambda_4 = w_2$ , we have  $\mu_1 = u_2, \mu_2 = v_2, \mu_3 = w_3, \mu_4 = w_1$ . But this application must not be allowed to blur the fundamental distinctions. The homography is a two-term relation between the variables  $\lambda$  and  $\mu$ , or between the element  $A$  which is identified by the value of  $\lambda$  and the element  $M$  which is identified by the value of  $\mu$ , and there is no need for the elements to be of the same kind: one may be a point and the other a line, or one may be a number and the other a conic. The involution-relation is not a relation between  $w_1$  and  $w_2$ , it is a three-term relation between the pair of numbers  $(w_1, w_2)$  and two other pairs of numbers, or between the pair of elements  $(W_1, W_2)$  and two other pairs of elements; the six elements are associated with six values of the same variable, and are therefore members of one and the same geometrical family. The involution-relation, as we see from (1) and (3), does not involve any arbitrary constants; it is one relation, which three pairs either do or do not satisfy. Whereas (8) specifies a particular homography, (5) identifies a particular involution-class.

It might seem that we make a double reduction when we pass from a general homography between distinct families to a symmetrical homography with coincident families. It is natural to conclude by asking if there is a theory of symmetrical homography which has the theory of involution as a case almost trivial in its extreme specialisation. The fact is that symmetry in a homography between variables  $\lambda, \mu$  can have no geometrical significance if the families with which the variables are associated are not identical. If  $A \neq 0$ , the general homography

$$A\lambda\mu + B_1\lambda + B_2\mu + C = 0$$

can be written

$$(A\lambda + B_2)(A\mu + B_1) = B_1B_2 - AC,$$

and if we are at liberty to use  $A\lambda + B_2$  as a new variable  $\lambda'$  in the one family and  $A\mu + B_1$  as a new variable  $\mu'$  in the other family, the relation between the variables takes the simple symmetrical form  $\lambda'\mu' = k$ . The symmetry is imposed in the algebra, and has no foundation in the geometry. If, however,  $\lambda$  and  $\mu$  are two values  $w_1, w_2$  of one variable  $w$  which is correlated with the elements of one family, any change of variable is a transformation of  $w$  and operates simultaneously on  $\lambda$  and  $\mu$ . A relation between  $\lambda$  and  $\mu$  which is not symmetrical can not yield a symmetrical relation by this simultaneous transformation, and if there is symmetry, the symmetry is intrinsic in the geometry. E. H. N.

## MATHEMATICAL NOTES

2343. *On even distribution of numbers.*

In his article on "The probability of a given error being exceeded in approximate computation" in *M.G.*, No. 308 (May, 1950), Mr. S. Inman seems to assume (Page 111) that the numbers to be multiplied are evenly distributed along the number-scale, so that the first significant figure is equally likely to be any of the range 1 to 9. This assumption, I believe, is not borne out by the facts.

First, what sort of numbers are we to imagine ourselves multiplying? Not sines of angles uniformly distributed along the angle-scale, for in these 9 predominates unduly. A similar objection might be found to apply to many trigonometrical functions. Not numbers chosen "at random" by Tom, Dick and Harry, because these gentlemen are notoriously incapable of choosing at random. I suggest that the numbers we should consider should be the numbers of arbitrary units in a quantity which someone might want to measure, acres of oats in counties, Joneses in towns and villages, labels in museums, or grammes in the weights of all the animals in a Zoo catalogue.

As an instance, imagine that skilled assessors have estimated the value of the material possessions of every inhabitant of our planet who has any, expressed in £ (or decimals of £1 in the case of the very poor). Then, on the theory of uniform distribution, 1/18 of these numbers begin with 10, 11, 12, 13 or 14; and 5/9 of them begin with 5, 6, 7, 8 or 9. Now, suppose you decide to reckon in shillings instead of pounds. 1/18 (instead of 1/9) of the numbers will now begin with 2, and 5/9 (instead of 1/9) will begin with 1. This seems to make the theory self-contradictory.

To my mind the only way out of this difficulty is to assume our numbers evenly distributed along a slide-rule, not a tape-measure, and all logarithms within the limits considered to be equally likely—a theory which is at any rate self-consistent if the range of numbers considered includes several powers of ten. On this assumption, if  $1 < a < 10$ , and  $1 < b < 10$ , the chance that  $ab < 10$  is not 1/5.79 but  $\frac{1}{2}$ .

As evidence of the occurrence of this kind of distribution in real life, it is stated in *M.G.*, No. 127 (Jan. 1917), p. 6, that of the populations of the 364 towns and London boroughs given in the Municipal Directory of England and Wales in *Whitaker's Almanack* for 1914, the percentage beginning with 1 is 29.7, which looks more like  $100 \log 2 / \log 10$  than  $100/9$ .

I suggest that anyone interested in testing this theory should reduce to pence his next hundred bills (apart from the regularly recurring ones such as Water Rate) and look at their first figures, or perhaps at the first figures in the answers in an Arithmetic book which are got by multiplication or division (excluding e.g. sums between 1/- and £1 expressed in shillings, which give an artificial predominance to 1).

W. HOPE-JONES.

2344. *On gauge constructions and a letter of Hjelmstev.*

1. It is well known that many elementary constructions usually performed by ruler and compass can be performed by the use of the gauge only.

Suppose we can join two points and find the cut, when it exists of two lines, and furthermore, perform the following:

A: An interval  $AB$ , called the "gauge", is given once for all; then a congruent interval can be cut off on any ray from any point of the ray.

We can then (see *Gazette*, XXIII, p. 465) without assuming the parallel axiom, draw perpendiculars, bisect angles and intervals, and reproduce congruent triangles anywhere.

2. If now we adjoin the Euclidean parallel axiom we can cut down A to the following and still perform these constructions :

B : An interval  $AB$  is given once for all ; then a congruent interval can be cut off on any ray from a given fixed point  $O$ .

This construction B gives us the points of meeting of a fixed gauge circle and any radius.

If we strengthen B to C :

C : The cuts of the fixed gauge circle and any line, if such cuts exist, can be found ; then we can perform all ruler and compass constructions.

3. Suppose now we do not assume the parallel axiom and we wish to get all ruler and compass constructions in the resulting non-Euclidean geometry. Neither B nor C is strong enough. Hjelmshlev (*Konstruktioner med normeret Lineal*, Tidsskrift, B. 1943) asserted that it was sufficient to assume

D : The cuts of any line, if they exist, and any circle of a fixed definite radius can be constructed (the gauge circle has still a fixed radius, but any centre). It is then possible to find the cuts of any line and any circle, or of two circles, if the cuts exist.

A little consideration shewed that his construction assumed that the following could be shown from congruence only, in two dimensions.

E : Suppose angles  $OAB, OBC, O'A'B', O'B'C'$  are right angles and that  $OA = O'A', AB = B'C', BC = A'B'$  then  $OC = O'C'$  (notice the twist in the hypothesis).

Hjelmshlev in his investigations assumes nothing about the intersection of lines, neither the Euclidean axiom, nor the axiom assumed by Hilbert for hyperbolic geometry. Nor does he make any continuity assumptions.

If E could be shewn the whole theory of radical axes and coaxial circles would follow easily in his general geometry.

4. After an unsuccessful attempt to prove E, I wrote to Hjelmshlev and the following is an extract from his letter of 28/11/1949 :

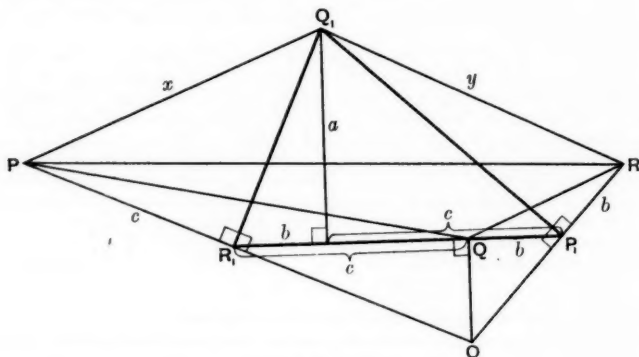


FIG. 1.

"... There are a great many apparently simple congruence-theorems the proof of which (by congruence alone) are really complicated. And this is one of them. But I will sketch you my own proof. In the triangle  $P_1Q_1R_1$ , let  $a$  be the altitude,  $b$  and  $c$  the pieces of the base. Prove that  $x=y$ . Now the triangles  $PQR$  and  $P_1Q_1R_1$  are orthologic, because  $PO \perp Q_1R_1, RO \perp P_1Q_1$ .

$QO \perp P_1R_1$ . Then the three perpendiculars from  $P_1, R_1, Q_1$  to  $QR, PQ, PR$  are concurrent. And as the first two of them bisect  $QR, PQ$ , the last one must bisect  $PR$ , which proves that  $x=y$ . . .

"As another interesting general congruence-theorem I state the following:

" $ABCD$  is a quadrilateral with three right angles  $A, B, C$ . The transversals  $BE$  and  $DF$  are equal. Then the angles  $\alpha$  and  $\beta$  are equal. This theorem contains the general construction of parallels.

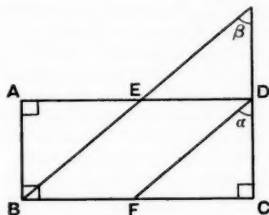


FIG. 2.

"And last, not least: Remember the median-theorem (*Kongruenzlehre* II p. 18)."

5. Some comments are needed. Hjelmslev takes for granted the results he has published in a series of papers on his theory of congruence in which nothing is assumed about the intersection of lines. In particular he uses his theory of ideal points and lines; the point  $O$  in Fig. 1 might be ideal. His work, in my view, is the most important on the foundations of Geometry in the plane since the days of Euclid. It must also be added that it includes the case when although two points sufficiently close can be joined by just one interval, this lies on an infinite number of lines. The interval fans out, as it were, at its ends.

The reference in the letter to the median theorem is to a proof, which he evidently prized, of the theorem that the medians of a triangle concur, using congruence in the plane only. It is a perplexing thing that such a theorem as this or the one proved in the letter should be so hard. Is there no criterion of "depth" in a given mathematical discipline?

This note is offered, because of the intrinsic value of the proof contained in the letter, and as a tribute to the memory of a distinguished geometer.

Hjelmslev died early in 1950 at the age of seventy-six. His papers on congruence will be found in *Mathematische Annalen* 64 (1907) and in the *Danske Videnskabernes Selskab*, in German, Vols. 8, 9, 19, 22, 25 (1929-1949).

H. G. FORDER.

# 2345. A lattice problem.

Some time ago I had a problem put to me as an amusing exercise by an eminent industrial research scientist, who is a much abler mathematician than I am. Neither he nor several members of his staff had found a general solution though I doubt if they could have put much effort into it. It illustrates very forcibly how a simple problem can appear difficult if presented or approached the wrong way. This is how it was put to me:

It is required to cover a country with a chain of broadcasting stations. If they all have the same power, and ignoring variations of terrain, they would ideally be placed at the intersections of a triangular lattice. If they are to send out different programmes, some listeners will get interference unless more than one frequency is used, and the more frequencies, the less the inter-

ference. Two frequencies will obviously give no improvement over one in a triangular lattice, but 3 will, and still better with 4, but 5 and 6 frequencies will give no improvement over 4, since it is impossible to design a pattern of frequency allocation of 5 or 6 frequencies which will give any greater separation between stations on the same frequency than can be obtained with four. By trial and error in working out the "patterns" they had obtained the series of numbers 3-4-7-9-13-16-19-21-25. Why this apparently disconnected series, and what is the general solution?

I rather naturally started on the same lines, by working out the "patterns". I soon found empirically a rather laborious method of working out the successive numbers of the series, and then quite accidentally noticed that there was a very simple formula for them, but I could see no proof for it, however. I therefore felt convinced that I had got my thoughts muddled, and dropped the problem. When I returned to it later, the solution at once appeared obvious, and the problem almost childish. Since several able people had failed to solve it, however, it would be interesting to know whether any readers of the *Gazette* see any difficulty before they read the next paragraph.

The problem as stated above is equivalent to finding the maximum spacing between stations on the same frequency, given the number of frequencies. If we reverse this by finding the minimum number of frequencies for a given spacing, the answer is simple. For, if  $d$  is the distance between adjacent stations, and  $D$  is the distance between stations on the same frequency, then for any given area of country the total number of stations and the number on any one frequency will be proportional to  $1/d^2$  and  $1/D^2$  respectively. Hence the number of frequencies is  $D^2/d^2$ , or if  $d$  is taken as unity, it is  $D^2$ . If we take two of the lattice lines as oblique axes, and  $(m, n)$  are the co-ordinates of the nearest station on the same frequency as that at the origin, then

$$D^2 = m^2 + mn + n^2.$$

The fact that  $m$  and  $n$  can only have integral values explains the apparently disconnected series for  $D^2$ .

E. V. NEWBERRY.

### 2346. A simple kinematical application of the cycloid.

Many readers will remember the case of the man who walks in a straight line whilst his dog tries to meet him by running at all times in the direction in which it sees its master. It is required to find the path of the dog. We deal here with a pursuer of a different kind. We assume that it can only move on a circle. If its initial position and direction, and those of the target which moves on a given straight line, are known, then we may ask what radius the circle ought to have, in order to ensure a hit of the pursuer on the target. The ratio of the pursuer's speed to that of the target will be denoted by  $k$ .

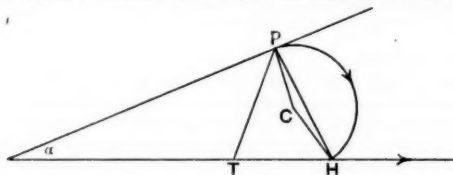


FIG. 1.

Consider Figure 1, where the initial positions of pursuer and target are respectively at  $P$  and  $T$ . The angle between the direction of the pursuer at  $P$  and of the target is  $\alpha$ . Let the pursuer move clockwise on a circle of radius  $r$  and centre  $C$ , and denote the point of hit by  $H$ . We have arc  $PH = k \cdot TH$ .

Now, reflect the arc  $PH$  in the straight line  $PH$  (Figure 2). The radii  $PC$  and  $HC$  are thereby reflected into  $PC'$  and  $HC'$ ;  $PCHC'$  is a rhombus and  $PC$ , and hence  $HC'$ , are both perpendicular to the initial direction of the tangent. It follows that the tangent in  $H$  to the arc  $HP$  with centre  $C'$  is parallel to this direction and that the chord on the target's path subtends an angle  $2\alpha$  at  $C'$ .

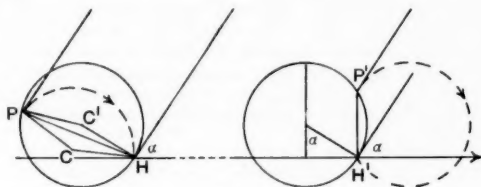


FIG. 2.

Imagine the circle with centre  $C'$  to move parallel to the target's path. At every position of the circle we mark on its perimeter the point  $P'$  such that the arc  $H'P'$  (Figure 2) is equal to  $k \cdot TH'$ . Then  $P'$  is again a point from which a hit can be obtained by a turn with radius  $r$ , provided that the initial direction is  $\alpha$ . (The pursuer's path is found by reflecting the circle in the line joining  $P'$  and  $H'$ .) It follows that the locus of all those points from which a hit will be obtained for given  $T$ ,  $r$ ,  $\alpha$  and  $k$  is traced by a point on a circle of radius  $r$ , rolling along a line which is parallel to the target's path at a distance  $r(1 - \cos \alpha)$  and (unless  $k = 1$ ) sliding at the same time. Alternatively, we can imagine a circle of radius  $r/k$  rolling without sliding on a line which is parallel to the target's path at a distance  $r(\cos \alpha - 1/k)$ , while a point at a distance  $r$  from the centre of the circle traces the locus. The latter is therefore a cycloid.

If  $k = 1$ , then we obtain the ordinary cycloid with cusps. If  $k$  is larger than 1, then a cycloid with loops and double points is obtained. We shall add some remarks about this type at a later stage.

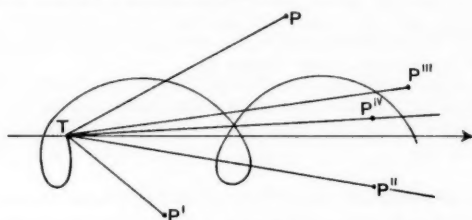


FIG. 3.

Let us now change  $r$  into  $R$ , say. All relations which we have so far derived remain valid if we imagine the plane transformed by extending all distances from  $T$  in the ratio  $R/r$ . The cycloid will be similarly transformed. This consideration leads immediately to the solution of our original problem of finding the radius for a suitable turn of the pursuer. We proceed as follows.

Draw the cycloid for  $T$ ,  $\alpha$ ,  $k$  and for radius 1 (see Figure 3, which has been drawn for  $\alpha = 90^\circ$ ). Draw also the initial position of the pursuer. If the line  $TP$ , protracted beyond  $P$  if necessary, intersects the cycloid in  $Q$ , then  $TP/TQ$  is the length of the required radius.



The method is due to Cunliffe (see L. E. Dickson. *History of the Theory of Numbers*, II, p. 204).

We have

$$4f^2 = 2(y^2 + z^2) - x^2 = y^2 + 2z^2. \dots\dots\dots(1)$$

Write  $d = x + y - z$  and  $m = 2f + x + 2y$ . Substitute for  $z$  and  $f$  in (1) then as  $x = y$ , we obtain

$$x(6m - 8d) = m^2 - 2d^2.$$

Thus the lengths of the sides  $x, y, z$  are

$$x = y = (m^2 - 2d^2)/2(3m - 4d), \quad z = (m - 2d)(m - d)/(3m - 4d). \dots\dots(2)$$

As  $z$  is integral,  $3m - 4d$  divides  $(m - 2d)(m - d)$ . A factor of  $3m - 4d$  that divides  $m - d$  also divides both  $m$  and  $2d$ . A factor of  $3m - 4d$  that divides  $m - d$  divides also both  $m$  and  $d$ . But  $m$  and  $d$  are relatively prime. For a factor common to them both would divide all of  $x, y, z, 2f, 2g$ , and there would be a smaller triangle similar to  $XYZ$  with  $x, y, z, 2f, 2g$  all integral. Thus

$$3m - 4d = \pm 1 \quad \text{or} \quad \pm 2. \dots\dots\dots(3)$$

Now, as  $x$  is integral 2 divides  $m$ . Hence,  $3m - 4d = \pm 1$  is impossible and we have  $m = 2p, 3p - 2d = \pm 1$ .

It follows that  $d$  is not divisible by 3 and there are two distinct cases,

$$d = 3k + 1, \quad p = 2k + 1, \dots\dots\dots(4)$$

or

$$d = 3k + 2, \quad p = 2k + 1. \dots\dots\dots(5)$$

(2) becomes either

$$x = y = \frac{1}{2}\{2(2k + 1)^2 - (3k + 1)^2\} \quad z = (-k)(k + 1) \dots\dots\dots(6)$$

or

$$x = y = -\frac{1}{2}\{2(2k + 1)^2 - (3k + 2)^2\} \quad z = k(k + 1). \dots\dots\dots(7)$$

As  $x, y, z$  are positive only the second alternative is possible. Thus

$$x = y = \frac{1}{2}(k^2 + 4k + 2) \quad z = k(k + 1) \dots\dots\dots(8)$$

and as  $x$  is integral  $k = 2l$ , so that

$$x = y = 2l^2 + 4l + 1 \quad z = 2l(2l + 1). \dots\dots\dots(9)$$

To obtain a genuine triangle from (9) it is necessary and sufficient that  $l$  be a positive integer.

From (9) the third median has length

$$h = \{(3l + 1)(4l + 1)(l + 1)\}^{\frac{1}{2}}. \dots\dots\dots(10)$$

If  $l$  is odd,  $l = 2s + 1$ , then  $h^2 = 4(3s + 2)(s + 1)(8s + 5)$ .  $3s + 2$  is prime to  $8s + 5$  and to  $s + 1$ . Thus, if  $h$  is an integer,  $3s + 2$  must be a square, which is impossible.

If  $l$  is even,  $l = 2s$ , then  $h^2 = (6s + 1)(8s + 1)(2s + 1)$ . The numbers  $2s + 1, 8s + 1$  are either relatively prime or have the common factor 3. They are both prime to  $6s + 1$ . Thus we have

$$2s + 1 = k_1^2, \quad 8s + 1 = k_2^2 \quad \text{or} \quad 2s + 1 = 3k_1^2, \quad 8s + 1 = 3k_2^2.$$

Hence  $4k_1^2 - k_2^2 = 1$  or 3. Thus  $2k_1 - k_2$  and  $2k_1 + k_2$  are divisors of 3. It is easily seen that this is impossible.

H. G. EGGLESTON.

2348. *A simple proof that all large integers are sums of at most eight cubes.*

It was shown by Landau that there exists a number  $n_0$  such that every integer  $n \geq n_0$  is representable as the sum of eight cubes of non-negative integers. Dickson\* obtained a numerical value for  $n_0$ , small enough (about

\* *Bull. Amer. Math. Soc.*, XLV (1939), 588-591.

$10^{15}$ ) to make it feasible to investigate the smaller integers systematically and to show that 23 and 239 are the only ones needing nine cubes. His proof is not self-contained and it involves heavy numerical work. The object of this note is to show that this theorem (which is, of course, not best possible and probably very far from it) is easier than has been realized, and need not involve heavy calculation.

2. Throughout this note, all letters denote non-negative integers.  $C_r$  denotes a sum of  $r$  (or fewer) cubes (of non-negative integers).  $\nu(n)$  (for  $n \neq 0$ ) is the exponent of the highest power of 5 that divides  $n$ . It is well-known that the congruence  $x^3 \equiv n \pmod{5}$  is soluble (uniquely) for every  $n$ . It is easy to deduce by induction on  $r$  that  $x^3 \equiv n \pmod{5^r}$  has always a unique solution if  $n \not\equiv 0 \pmod{5}$ , that is, if  $\nu(n) = 0$ . This congruence is also soluble, though not uniquely, for  $\nu(n) > 0$ , provided that  $\nu(n)$  either divides by 3 or is not less than  $r$ .

3. We now show that  $N$  is  $C_6$  if there exists an  $m$  such that

$$m \text{ is prime to six, } \dots\dots\dots (1)$$

$$\frac{3}{2}m^3 < N < \frac{3}{2}m^3, \dots\dots\dots (2)$$

$$N \equiv 3m \pmod{6m}. \dots\dots\dots (3)$$

To prove this, note that by (2) and (3),

$$8N = 6m^3 + 6mk, \quad 0 < k < m^2.$$

Hence  $6mk \equiv 8N - 6m^3 \equiv 24m - 6m \equiv 18m \pmod{48m}$ , whence by (1)  $k \equiv 3 \pmod{8}$ . Using the classical three-square theorem,  $k$  is the sum of three odd squares. Writing  $k = x_1^2 + x_2^2 + x_3^2$ , we have

$$\begin{aligned} 8N &= 6m^3 + 6m(x_1^2 + x_2^2 + x_3^2) \\ &= \sum_{i=1}^3 \{(m+x_i)^3 + (m-x_i)^3\}, \end{aligned}$$

identically. Since each  $x_i \leq k^{1/2} < m$ ,  $8N$  is the sum of six positive, even cubes, whence the result.

4. If we put  $n = x^3 - y^3$  for  $N$ , we see (breaking (3) up into two congruences, as we may by (1), and noting that  $x^3 \equiv x \pmod{6}$ ), that  $n$  is  $C_6$  if for suitable  $m$  we can solve

$$\frac{3}{2}m^3 < n - x^3 - y^3 < \frac{3}{2}m^3, \dots\dots\dots (4)$$

$$x^3 + y^3 \equiv n \pmod{m}, \dots\dots\dots (5)$$

$$x + y \equiv n + 3 \pmod{6}. \dots\dots\dots (6)$$

If we put  $m = 5^r$ , with  $r$  defined by

$$5^{3r} \leq n < 5^{3(r+1)}, \dots\dots\dots (7)$$

we shall have  $r \geq 10$  if  $n \geq 5^{30}$ . (4) will hold if

$$0 \leq x < \frac{1}{2}m, \dots\dots\dots (8)$$

and

$$n - \frac{7}{8}m^3 > y^3 \geq \max(n - \frac{3}{2}m^3, 0). \dots\dots\dots (9)$$

It is readily verified that even in the worst case, when

$$n = 5^{3r+3} - 1 = 125m^3 - 1,$$

(9) defines an interval of length greater than  $m/120$ .

5. It will be clear from the next paragraph that (with  $m = 5^r$ ), equation (5) always has a solution with  $x \equiv y \equiv 0 \pmod{5^r}$  if  $n \equiv 0 \pmod{5^{3r}}$ . In case  $\nu(n) \geq 6$ , we can satisfy (5), (6), (8), (9) if  $r \geq 6$ ; for on putting  $x, y = 25X, 25Y$  we have a congruence to modulus  $6 \cdot 5^{-6} \cdot m$ , while the intervals in which

$X, Y$  must lie are of lengths at least  $6 \cdot 5^{-s} \cdot m, 5^{-s} \cdot m$  respectively. We next show that we can solve (5), (6), (8), (9) for  $r \geq 10, \nu(n) \leq 2$ .

6. If  $\nu(n) = 0$ , we can solve

$$(1 + \lambda^3)X^3 \equiv n \pmod{5^{r-s}}, \dots\dots\dots(10)$$

with  $X \not\equiv 0 \pmod{5}$ , for any  $\lambda \not\equiv -1 \pmod{5}$ , that is, for any  $\lambda$  with  $\nu(1 + \lambda^3) = 0$ . We may note that as  $1 - \lambda + \lambda^2 \equiv 0 \pmod{5}$  has no solution,

$$\nu(1 + \lambda^3) = \nu(1 + \lambda) + \nu(1 - \lambda + \lambda^2) = \nu(1 + \lambda).$$

If  $\nu(n) = 1$ , (10) is soluble if  $\nu(1 + \lambda) = 1$ , and if  $\nu(n) = 2$ , (10) is soluble if  $\nu(1 + \lambda) = 2$ . Evidently we may choose  $\lambda$  so that  $\nu(1 + \lambda) = 0, 1$  or  $2$ , as required, and

$$\lambda^3 \equiv 1201 \text{ or } 1231 \pmod{5^s}. \dots\dots\dots(11)$$

7. We can therefore satisfy (for any  $n \not\equiv 0 \pmod{125}$ )

$$X^3 + Y^3 \equiv n \pmod{5^{r-s}}, \dots\dots\dots(12)$$

$$Y \equiv \lambda X \pmod{5^{r-s}}, \dots\dots\dots(13)$$

$$XY \not\equiv 0 \pmod{5}, \dots\dots\dots(14)$$

$$0 \leq X, Y < 5^{r-s}. \dots\dots\dots(15)$$

If we now put  $x, y = X + 5^{r-s}u, Y + 5^{r-s}v$ , (5) becomes a linear congruence in  $u, v$  if  $2(r-5) \geq r$ , that is, if  $r \geq 10$ . A factor  $5^{r-s}$  can be cancelled out, and then (5) reduces, using (13), (14), to

$$u + \lambda^2 v \equiv M \pmod{5^s},$$

for some  $M$ . We may suppose  $0 \leq M < 6 \times 5^s = 18,750$  and  $M \equiv n + 3 \pmod{6}$ , and then using (11), (5) and (6) together become

$$u + \mu v \equiv M \pmod{18,750}, \dots\dots\dots(16)$$

with  $\mu = 1201$  or  $1231$ .

8. Now (16) is to be solved subject to  $0 \leq u \leq 1561$ , which with (15) ensures that (8) holds, and to (9). (9) permits at least  $[5^s/120] = 26$  consecutive integral values of  $v$ , which by a transformation we may suppose to be the values  $0, 1, \dots, 25$ . Clearly the solution  $v = [M/\mu] \leq 15$  of (16) gives what is required.

9. We have thus shown that  $n$  is  $C_8$  if  $n \geq 5^{30}$ , unless  $\nu(n) = 3, 4$ , or  $5$ . In these three cases we could modify the above argument to obtain the same (or a better) result; but noting that  $n$  is  $C_8$  if  $n/125$  is integral and  $C_8$ , it follows that in all cases  $n$  is  $C_8$  if  $n \geq 5^{33}$ . This is the theorem, with  $n_0 = 5^{33} < 2 \times 10^{33}$ . By choosing  $m$  differently, smaller integers down to about  $10^{12}$  can be dealt with.

G. L. WATSON.

# 2349. On Note 2045 : an algebraic identity.

The identity discussed by Mr. Krishnaswami Ayyangar admits of a simple proof by the methods of algebraical geometry. Transforming to polar co-ordinates in the plane  $x + y + z = 0$  by the substitution

$$x = \frac{2r}{\sqrt{6}} \cos \theta, \quad y = -\frac{r}{\sqrt{6}} \cos \theta + \frac{r}{\sqrt{2}} \sin \theta, \quad z = -\frac{r}{\sqrt{6}} \cos \theta - \frac{r}{\sqrt{2}} \sin \theta,$$

we have

$$3\sqrt{6} \cdot xyz = r^3 \cos 3\theta, \quad \sqrt{2}(x-y)(y-z)(z-x) = -r^3 \sin 3\theta.$$

If  $\rho, \phi$  are the polar coordinates of  $(a, b, c)$  then the scalar product

$$ax + by + cz = r\rho \cos(\theta - \phi).$$

Consequently

$$\begin{aligned} 4(ax + by + cz)^3 - 3(ax + by + cz)(a^2 + b^2 + c^2)(x^2 + y^2 + z^2) - 54abcxyz \\ - 2(x - y)(y - z)(z - x)(a - b)(b - c)(c - a) \\ = r^3 \rho^3 (4 \cos^3 (\theta - \phi) - 3 \cos (\theta - \phi) - \cos 3\theta \cos 3\phi - \sin 3\theta \sin 3\phi) \\ = r^3 \rho^3 (\cos 3(\theta - \phi) - \cos 3(\theta - \phi)) = 0. \end{aligned}$$

H. J. GODWIN.

2350. On Note 2045 : an algebraic identity.

Note 2045 (Prof. Krishnaswami Ayyangar) quotes and proves that given

$$a + b + c = x + y + z = 0,$$

then

$$\begin{aligned} 4(ax + by + cz)^3 - 3(ax + by + cz)(a^2 + b^2 + c^2)(x^2 + y^2 + z^2) - 54abcxyz \\ = 2(b - c)(c - a)(a - b)(y - z)(z - x)(x - y). \end{aligned} \quad \dots\dots\dots(i)$$

An alternative proof may be given as follows.

Let  $u = ax$ ,  $v = by$ ,  $w = cz$ ,

$$\theta = u + v + w, \quad \phi = uv + vw + wu, \quad \psi = uvw.$$

Now,

$$(b + c)(y + z) = ax,$$

so that

$$bz + cy = ax - by - cz = 2u - \theta.$$

Hence

$$(b - c)(y - z) = by + cz - (2u - \theta) = 2\theta - 3u.$$

Accordingly, the right-hand side of (i) can be replaced by

$$\begin{aligned} 2\Pi\{(b - c)(y - z)\} &= 2\Pi(2\theta - 3u) \\ &= 2\{8\theta^3 - 12\theta^2 \Sigma u + 18\theta \Sigma uv - 27uvw\} \\ &= -8\theta^3 + 36\theta\phi - 54\psi. \end{aligned} \quad \dots\dots\dots(ii)$$

In order to reduce the left-hand side of (i) we note that

$$\begin{aligned} \Sigma a^2 \Sigma x^2 &= (\Sigma ax)^2 + \Sigma (bz - cy)^2 \\ &= \theta^2 + \Sigma (bz + cy)^2 - 4 \Sigma bcyz \\ &= \theta^2 + \Sigma (\theta - 2u)^2 - 4\phi \\ &= \theta^2 + 3\theta^2 - 4\theta \Sigma u + 4 \Sigma u^2 - 4\phi \\ &= 4\theta^2 - 4\theta^2 + 4(\theta^2 - 2\phi) - 4\phi \\ &= 4\theta^2 - 12\phi. \end{aligned}$$

Hence the left-hand side of (i) becomes

$$4\theta^3 - 12\theta^3 + 36\theta\phi - 54\psi = -8\theta^3 + 36\theta\phi - 54\psi,$$

as before.

B. E. LAWRENCE.

2351. *Pyramidal indentation of a sphere.*

This note gives a method of determining the area of the sides of a square pyramidal indentation in a sphere of radius  $R$ , given that the angle between the normals to opposite faces is  $2\theta$  and that the diagonal distance measured across the opening is  $d$ .

In the figure, the area  $BDCB$  represents one face of the indentation symmetrical with respect to axes at  $O$ , the  $z$  axis being the axis of symmetry.

Let  $EB = EC = \frac{1}{2}d$ ,  $OC = R$ , radius of arc  $BC = r$ , and  $\angle ADB = \angle ADC = \alpha$ . Then

$$\begin{aligned} \rho_B &= [0, \tfrac{1}{2}d, \sqrt{(R^2 - \tfrac{1}{4}d^2)}], \\ \rho_D &= [0, 0, \sqrt{(R^2 - \tfrac{1}{4}d^2)} - \tfrac{1}{2}d \cot \alpha], \\ \rho_C &= [\tfrac{1}{2}d, 0, \sqrt{(R^2 - \tfrac{1}{4}d^2)}]. \end{aligned}$$



When  $d^2 \ll R^2$  we can neglect powers of  $d$  and hence

$$r \approx \sqrt{(R^2 \cos^2 \theta + \frac{1}{4} \cdot dR \sin 2\theta)}. \quad (6)$$

The area of the side of the indentation is the area of the triangle  $BDC$  plus the area of the segment  $BCB$ . The vector area of the triangle  $BDC$  is

$$\frac{1}{2}[(\rho_B - \rho_D) \times (\rho_C - \rho_D)] \quad (7)$$

and so the area of  $BDC$  is the modulus of the vector given by equation (7), namely,

$$\frac{1}{2}d^2 \operatorname{cosec}^2 \theta. \quad (8)$$

The area of the segment of radius  $r$  and chord length  $\frac{1}{2}d\sqrt{2}$  is given as

$$\frac{1}{2}rs - \frac{1}{4}d\sqrt{2} \cdot \sqrt{(r^2 - \frac{1}{8}d^2)}, \quad (9)$$

where  $s$  is the arc length of the segment. The length of  $s$  may be expressed in terms of the chord very simply as

$$s\sqrt{2} = d + (d^3/48r^2) + (3d^5/2560r^4) + \dots \quad (10)$$

Substituting for  $s$  in equation (9), we have the area of the segment as

$$0.0295(d^3/r) + 0.0011(d^5/r^3). \quad (11)$$

Thus the area of the side is

$$\frac{1}{2}d^2 \operatorname{cosec}^2 \theta + 0.0295(d^3/r) + 0.0011(d^5/r^3), \quad (12)$$

and the total area  $A$  of the indentation sides is given by

$$A = 0.5d^2 \operatorname{cosec}^2 \theta + 0.1178(d^3/r) + 0.0044(d^5/r^3). \quad (13)$$

The area may thus be found to any degree of accuracy by extending the terms in the series, and substitution of the value of  $r$  found from equations (5) or (6) in equation (13) gives the required result. One application occurs in practice, for example, penetration hardness testing with a pyramidal diamond as used with a Vickers machine. In this case the value of  $2\theta$  is restricted to  $136^\circ$  and

$$A \approx 0.5393d^2 + d^3(10.104R^2 + 17.683dR)^{-1/2} + d^5(5.210R^2 + 9.120dR)^{-3/2}, \dots \quad (14)$$

where the approximate value for  $r$  has been substituted in equation (13).

N. J. C. PERES

### 2352. Approximations to $\log_e \left( \frac{x+1}{x-1} \right)$ .

#### 1. The continued fraction development.

Let  $R(x) = \log_e \left( \frac{x+1}{x-1} \right)$ , and  $x$  real with  $|x| > 1$ . Then the continued fraction for  $R(x)$  is given by

$$R(x) = \frac{2}{x} - \frac{\frac{1}{2}}{\frac{3x}{2} - \frac{\frac{3}{2}}{\frac{5x}{2} - \frac{\frac{5}{2}}{\frac{7x}{2} - \frac{\frac{7}{2}}{\frac{9x}{2} - \frac{\frac{9}{2}}{\frac{11x}{2}}}}} \dots \quad (1)$$

We shall write  $L_s(x) = N_s(x)/D_s(x)$  for the  $s$ th convergent of (1) and use  $R, L_s, N_s, D_s$  for  $R(x)$ , etc.

It follows from (1) that  $N_s$  and  $D_s$  satisfy the recurrence relation

$$sW(s) = (2s-1)x(W(s-1) - (s-1)W(s-2)), \quad s = 2, 3, \dots, \dots \quad (2)$$

with

$$\begin{cases} N_0 = 0, & N_1 = 2 \\ D_0 = 1, & D_1 = x. \end{cases}$$

For example,

$$\begin{aligned} N_2 &= 3x, & N_3 &= 5x^2 - \frac{4}{3}, & N_4 &= \frac{1}{12}(105x^3 - 55x), \\ D_2 &= \frac{1}{2}(3x^2 - 1), & D_3 &= \frac{1}{2}(5x^3 - 3x), & D_4 &= \frac{1}{8}(35x^4 - 30x^2 + 3). \end{aligned}$$

Indeed, it is well known that  $D_s$  is the  $s$ th Legendre polynomial, and may be expressed in the form of Laplace's first integral

$$D_s = \frac{1}{\pi} \int_0^\pi [x + \cos \theta \sqrt{x^2 - 1}]^s d\theta, \quad |x| > 1. \quad (3)$$

It is evident from (3) that (a)  $D_s > 0$  if  $x > 1$ , and if  $x < -1$  provided  $s$  is even; (b)  $D_s < 0$  if  $x < -1$  and  $s$  is odd.

For later use we note that, using (3) in Schwarz's inequality,

$$\left| \begin{matrix} \int a^2 dx & \int ab dx \\ \int ba dx & \int b^2 dx \end{matrix} \right| > 0$$

(assuming  $a$  and  $b$  are linearly independent), we have

$$\left| \begin{matrix} D_s & D_{s+1} \\ D_{s+1} & D_{s+2} \end{matrix} \right| > 0 \quad \text{for } |x| > 1. \quad (4)$$

Again, since  $D_s(1) = 1$ , and  $D_s(-1) = (-1)^s$  it follows that  $D_s D_{s+2} - D_{s+1}^2$  has  $x^2 - 1$  as a factor.

## 2. The definite integral for $RD_s - N_s$ .

Let  $\Phi_s(y) = yD_s - N_s$ .

Then, by a result due to Gauss (Werke, III, pp. 186-189),

$$\begin{aligned} \Phi_s(R) &= RD_s - N_s \quad (5) \\ &= \frac{1}{2^s} \int_{-1}^1 \frac{(1-t^2)^s dt}{(x-t)^{s+1}} = Q_s(x), \end{aligned}$$

where  $Q_s(x)$  is Legendre's function of the second kind. Using (5) in Schwarz's inequality, we have

$$f_s(R) = \left| \begin{matrix} \Phi_s(R) & \Phi_{s+1}(R) \\ \Phi_{s+1}(R) & \Phi_{s+2}(R) \end{matrix} \right| > 0 \quad (6)$$

so that if  $R_s$  is the greater root of  $f_s(R) = 0$ , then since  $D_s D_{s+2} - D_{s+1}^2 > 0$ , we have

$$R_s = \frac{D_s N_{s+2} + D_{s+2} N_s - 2D_{s+1} N_{s+1} + \sqrt{(v_s^2 - 4u_s w_s)}}{2(D_s D_{s+2} - D_{s+1}^2)}, \quad (7)$$

where  $f_s(R) = u_s R^2 - v_s R + w_s$  with

$$u_s = D_s D_{s+2} - D_{s+1}^2, \quad v_s = D_s N_{s+2} + D_{s+2} N_s - 2D_{s+1} N_{s+1}, \quad w_s = N_s N_{s+2} - N_{s+1}^2.$$

Now, by using (2), we find the determinant relations

$$\left. \begin{aligned} L_s - L_{s-1} &= \frac{1}{s D_{s-1} D_s} \\ L_s - L_{s-2} &= \frac{(2s-1)x}{s(s-1) D_{s-2} D_s} \end{aligned} \right\}, \quad (8)$$

so that for  $x > 1$ ,

$$L_0 < L_1 < L_2 \dots < R. \quad (9)$$

Hence

$$f_s(L_s) < 0, \quad f_s(L_{s+1}) < 0, \quad f_s(L_{s+2}) < 0, \quad f_s(R_s) = 0, \quad f_s(R) > 0$$

so that

$$L_s < L_{s+1} < L_{s+2} \dots < R_s < R, \quad x < 1,$$

where, using (8) on  $v_s^2 - 4u_s w_s$ ,

$$R_s = \frac{(s+2)(s+1)(D_s N_{s+2} + D_{s+2} N_s - 2D_{s+1} N_{s+1}) + 2\sqrt{(2s+3)^2 x^2 - 4(s+2)(s+1)}}{(s+2)(s+1)(D_s D_{s+2} - D_{s+1}^2)} \dots\dots(10)$$

3. The sequence  $\{R_s\}$  is monotonic increasing.

Since  $\Phi_s(y) = yD_s - N_s$  it is evident that  $\Phi_s(y)$  follows the recurrence relation (2) with  $\Phi_0(y) = y$ ,  $\Phi_1(y) = xy - 2$ . Using the recurrence relation on the second row of

$$| \Phi_s(y), \Phi_{s+1}(y) |,$$

we find

$$\begin{aligned} f_s(y) &= \frac{s}{s+1} f_{s-1}(y) + \Phi_s(y) \{A_s \Phi_{s+1}(y) - B_s \Phi_s(y)\} \dots\dots\dots(11) \\ &= \frac{s}{s+1} f_{s-1}(y) + \Phi_s(y) \Psi_s(y) \text{ say,} \end{aligned}$$

where

$$A_s = \frac{(2s+3)^2}{(s+2)^2} x - \frac{(2s+1)x}{(s+1)} - \frac{(s+1)}{(s+2)}, \quad B_s = \frac{(2s+3)(s+1)x}{(s+2)^2} - \frac{s}{s+1}.$$

But

$$\Phi_s(R_{s-1}) = D_s(R_{s-1} - L_s) > 0 \quad \text{for } x > 1$$

so that  $f_s(R_{s-1})$  has the same sign as  $\Psi_s(R_{s-1})$ . But

$$A_s = \frac{(2s^3 + 7s^2 + 9s + 5)x}{(s+1)(s+2)^2} - \frac{(s+1)}{(s+2)}$$

and so  $A_s > 0$  for  $x > 1$  since  $A_s > 0$  when  $x = 1$ . Similarly  $B_s > 0$  when  $x > 1$ .

Hence

$$\Psi_s(L_s) < 0, \quad \Psi_s(L_{s+1}) < 0 \quad \text{for } x > 1.$$

Moreover

$$\Psi_s(R) = A_s \Phi_{s+1}(R) - B_s \Phi_s(R)$$

and using (5)

$$\begin{aligned} \Psi_s(R) &= \frac{A_s}{2^{s+1}} \int_{-1}^1 \frac{(1-t^2)^{s+1} dt}{(x-t)^{s+2}} - \frac{B_s}{2^s} \int_{-1}^1 \frac{(1-t^2)^s dt}{(x-t)^{s+1}} \\ &= \frac{1}{2^s} \int_{-1}^1 (A_s t - B_s) \frac{(1-t^2)^s dt}{(x-t)^{s+1}} \end{aligned}$$

after integration by parts. But  $A_s - B_s = (1-x)/(s+1)(s+2) < 0$  for  $x > 1$ . Hence  $\Psi_s(R) < 0$ . But  $\Psi_s(L_{s+1}) < 0$  and  $L_{s+1} < R_{s-1} < R$ . Hence  $\Psi_s(R_{s-1}) < 0$  and so  $f_s(R_{s-1}) < 0$ . Thus we have

$$L_s < L_{s+1} \dots < R_{s-1} \dots < R$$

and  $f_s(L_s) < 0$ ,  $f_s(L_{s+1}) < 0$ ,  $f_s(R_{s+1}) < 0$ ,  $f_s(R_s) = 0$ ,  $f_s(R) > 0$  so that  $R_s > R_{s-1}$  and the sequence  $\{R_s\}$  is monotonic increasing, and has the limit  $R$ . In particular, for  $x > 1$ ,

$$R_0 = \frac{-x + \sqrt{(9x^2 - 8)}}{x^2 - 1} < \log_e \left( \frac{x+1}{x-1} \right),$$

$$R_0 < R_1 = \frac{2[3x^3 - 4x + \sqrt{(25x^2 - 24)}]}{3(x^2 - 1)(x^2 + 1)} < \log_e \left( \frac{x+1}{x-1} \right), \dots\dots\dots(13)$$

$$R_1 < R_2 = \frac{2[15x^5 - 10x^3 - 7x + 2\sqrt{(49x^2 - 48)}]}{3(x^2 - 1)(5x^2 + 3)} < \log_e \left( \frac{x+1}{x-1} \right),$$

etc. The case when  $x < -1$  can be treated similarly and we have only to reverse the inequalities in (13) and change the sign of the square root term.

L. R. SHENTON.

2353. *Certain statistical kinematic identities.*

Consider an assembly of  $N$  objects with which two measurable properties  $x$  and  $u$  are associated. By the usual definitions,

$$\bar{x} = \Sigma x_i / N, \quad \bar{u} = \Sigma u_i / N, \dots\dots\dots(1)$$

$$\mu_{n,m} = \Sigma (x_i - \bar{x})^n (u_i - \bar{u})^m / N, \dots\dots\dots(2)$$

and

$$\sigma_x^2 = \mu_{2,0}, \quad \sigma_u^2 = \mu_{0,2}, \quad \mu = \mu_{1,1} = \rho \sigma_x \sigma_u \dots\dots\dots(3)$$

The last relation defines the correlation coefficient  $\rho$  and it can be shown that

$$-1 < \rho < 1.$$

Let the "objects" be particles or moving points, and let the "properties" be "position" and "velocity" in one dimension, respectively, so that

$$\frac{dx_i}{dt} = u_i \dots\dots\dots(4)$$

Adding for all  $i$ , it follows from the definitions (1) above that

$$\dot{\bar{x}} = \bar{u} \dots\dots\dots(5)$$

Differentiating (2) and using (4), one finds similarly,

$$\frac{d\mu_{n,m}}{dt} = n\mu_{n-1,m+1} + m\Sigma (x_i - \bar{x})^n (u_i - \bar{u})^{m-1} (\dot{u}_i - \dot{\bar{u}}) / N, \dots\dots\dots(6)$$

and

$$\frac{d\mu_{n,0}}{dt} = n\mu_{n-1,1} \dots\dots\dots(7)$$

In particular,

$$\frac{d\sigma_x^2}{dt} = 2\mu \quad \text{or} \quad \dot{\sigma}_x = \rho \sigma_u \dots\dots\dots(8)$$

Although derived here for a discrete distribution, these relations hold generally: for a bivariate continuous probability distribution  $f = f(x, u; t)$ , where

$$\bar{x} = \iint x f(x, u; t) dx du \dots\dots\dots(9)$$

etc., they follow similarly from the definitions and by use of the continuity relation necessarily satisfied by  $f$ , viz.

$$\frac{\partial f}{\partial t} + u \frac{\partial f}{\partial x} + \frac{\partial}{\partial u} a f = 0 \dots\dots\dots(10)$$

(Liouville's theorem), where  $a$  refers to the acceleration.

If  $\sigma_x$  and  $\sigma_u$  are interpreted as uncertainties in position and velocity, respectively, relation (8) in its second form is of particular interest. Further simple relations follow from (6)–(8) for the initial values of the first few derivatives of some of the  $\mu_{mn}$  under special initial conditions, e.g. certainty in initial position; they have some bearing on the assumptions underlying the "approximate" (diffusion) and the so-called "accurate" (kinetic) theory of Brownian motion (cf. Klein, Ph.D. thesis, London 1951), and they may have applications in other fields.

G. KLEIN.

2354. *The deformed circular ring.*

1. A thin wire bent in the form of a circle of radius  $r$ , is suspended in a vertical plane from its highest point and deformed by a force  $mg$  applied vertically downwards at the lowest point  $B$ , which in consequence descends a distance  $d$ .

Calthrop and Miller\* measured  $d$  for wires of known dimensions and materials and calculated  $Y$ , Young's modulus, for them. They gave, without proof, the formula  $Y = 19 mgr^2/a^4d$  where  $a$  is the radius of cross-section of the wire (assumed circular) and refer to the work of Sucksmith,† who quotes, without proof or reference, a similar result. Sucksmith applied an inhomogeneous magnetic field vertically downwards at  $B$  instead of a weight and calculated the magnetic susceptibility of the material of the wire. The proof follows.

2. The deformed wire is symmetrical about rectangular axes  $Ox$ ,  $Oy$ , the latter vertical and passing through the point of support  $A$ .  $P(x, y)$  is any point on the arc  $AB$  where the bending moment is  $M$  and the tangent makes

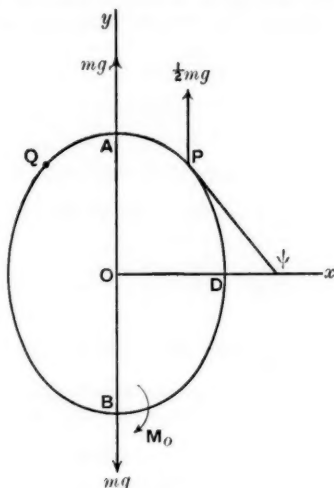


FIG.

an angle  $\psi$  with the axis of  $x$ , and  $Q$  is the image of  $P$  in  $Oy$ . For the equilibrium of  $PQB$ , vertically it is seen that the force exerted on  $BP$  by  $AP$  is  $\frac{1}{2}mg$  vertically upwards, the weight of the wire being neglected. Taking moments about  $B$  for  $BP$ ,

$$\frac{1}{2}mgx + M = M_0, \dots\dots\dots(1)$$

where  $M_0$  is the bending moment at  $B$ . Now  $M = YI/\rho$ , where  $I$  is the second moment of area of the cross-section of the wire about a line of symmetry in its plane perpendicular to the radius of curvature  $\rho$  of the curve  $APB$  at  $P$ .

Differentiating (1) with respect to  $\psi$ ,

$$\frac{1}{2}mg \frac{dx}{ds} \frac{ds}{d\psi} = \frac{YI}{\rho^2} \frac{d\rho}{d\psi} \dots\dots\dots(1a)$$

and since  $dx/ds = \cos \psi$ ,  $ds/d\psi = \rho$ , we have

$$\frac{1}{2}mg \cos \psi \, d\psi = YI \, d\rho/\rho^3. \dots\dots\dots(1b)$$

\* Calthrop and Miller, *Amer. Phys. Teacher*, 3, 131-2 (1935).

† Sucksmith, *Phil. Mag.*, 8, 158 (1929).

Integrating (1b),

$$mg \sin \psi = YI (R_1^{-2} - \rho^{-2}) \dots\dots\dots(2)$$

and

$$mg = YI (R_1^{-2} - R_2^{-2}),$$

where  $R_1$  and  $R_2$  are the radii of curvature at points where  $\psi$  is 0 and  $\frac{1}{2}\pi$  respectively.

3. Writing  $mgR_1^2/YI = k$  which must be kept as small as  $2 \cdot 10^{-1}$  by use of small values of  $m$  for the accuracy of the following, (2) gives

$$1/\rho = (1/R_1) \cdot \sqrt{(1 - k \sin \psi)}. \dots\dots\dots(3)$$

Hence the arc  $s$  from  $B$  to  $P$  is given by

$$R_1 \int_0^\psi \frac{d\psi}{\sqrt{(1 - k \sin \psi)}}.$$

The force  $\frac{1}{2}mg$  at  $P$  is the resultant of shear and tension at  $P$  so that the tension is greatest at  $D$ , where it is  $\frac{1}{2}mg$ . Such a tension would extend a straight wire of length  $BD$  an amount  $\pi mgr/4SY$ , where  $S$  is the cross-sectional area of the wire and in ordinary experiments this is of the order  $10^{-5}$  to  $10^{-4}$  cms., negligible compared with  $d$ , which is about 1 cm. Thus

$$\frac{1}{2}\pi r = R_1 \int_0^{\pi/2} \frac{d\psi}{\sqrt{(1 - k \sin \psi)}}. \dots\dots\dots(4)$$

Since the shape of the curve admits of values of  $\psi$  to and beyond  $\frac{1}{2}\pi$ , it is seen from (4) that  $k \leq 1$ . For  $k < 1$  the integrand in (4) can be expanded by the binomial theorem and it is justifiable to integrate term by term. Thus

$$\frac{1}{2}\pi r = R_1 \int_0^{\pi/2} (1 + \frac{1}{2}k \sin \psi + \dots) d\psi \simeq \frac{1}{2}R_1 (\pi + k)$$

or  $r \simeq R_1 (1 + k/\pi). \dots\dots\dots(5)$

From (2), using  $dy/ds = \sin \psi$ ,

$$k dy/d\psi = \rho^{-2} R_1^2/\rho, \dots\dots\dots(6)$$

whence, by (3),

$$ky/R_1 = \int_0^\psi \frac{k \sin \psi d\psi}{\sqrt{(1 - k \sin \psi)}}. \dots\dots\dots(7)$$

Proceeding as with (4) and integrating from 0 to  $\frac{1}{2}\pi$ , where the  $y$  values are  $-(r + \frac{1}{2}d)$ , 0, respectively, we have

$$k(r + \frac{1}{2}d) = R_1 \int_0^{\pi/2} (k \sin \psi + \frac{1}{2}k^2 \sin^2 \psi + \dots) d\psi$$

or  $r + \frac{1}{2}d \simeq R_1 (1 + \frac{1}{8}k\pi). \dots\dots\dots(8)$

This, with (5), gives  $1 + d/2r = (1 + \frac{1}{8}k\pi)/(1 + k/\pi)$  and for small values of  $k$  this gives  $d/r = k(\pi^2 - 8)/4\pi$ , so that

$$YI/mgR_1^2 (= 1/k) = (\pi^2 - 8)r/4\pi d. \dots\dots\dots(9)$$

Hence, on using (5) to eliminate  $R_1$ , we have

$$Y \simeq \frac{(\pi^2 - 8)mgr^3}{4\pi dI} \left(1 + \frac{k}{\pi}\right)^{-2},$$

or  $Y \simeq \frac{mgr^3(\pi^2 - 8)}{4\pi dI} - \frac{2mgr^2}{\pi I}. \dots\dots\dots(10)$

For a wire of circular cross-section and radius  $a$ ,  $I = \frac{1}{4}\pi a^4$  and the result follows since the second term in (10) is negligible compared with the first.

When  $k$  is very small, the above working gives

$$s \doteq R_1 (\phi + \frac{1}{2}k \sin \phi),$$

$$y \doteq R_1 (\sin \phi + \frac{1}{4}k\phi + \frac{1}{4}k \sin \phi \cos \phi),$$

$$x \doteq R_1 (\cos \phi + \frac{1}{4}k \cos^2 \phi),$$

where  $\phi = \psi - \frac{1}{2}\pi$  and  $s$  is measured from the point at which  $\phi$  is zero.

H. W. HASKEY.

### 2355. *A dielectric cylinder.*

1. If a circular cylinder of radius  $a$ , filled with homogeneous dielectric of specific inductive capacity  $k$ , is placed at the origin of coordinates in a two-dimensional electrostatic field whose complex potential is  $f(z)$  in air, having no singularities inside or on  $r = a$ , then the complex potentials inside and outside the cylinder are respectively

$$\Omega_i = \frac{2}{1+k} f(z) \quad \text{and} \quad \Omega_o = f(z) + \frac{1-k}{1+k} \bar{f}\left(\frac{a^2}{z}\right).$$

Since all the singularities of  $f(z)$  are outside  $r = a$ , then on or near  $r = a$  the potential function  $\phi(r, \theta)$ , the real part of  $f(z)$ , may be expanded in the form

$$\phi(r, \theta) = \sum_1^p A_n r^n T_n$$

where  $T_n$  is the solution of  $d^2y/d\theta^2 + n^2y = 0$ .

The potentials inside and outside the dielectric may be taken to be

$$\phi_i = \sum_1^p B_n r^n T_n \quad \text{and} \quad \phi_o = \phi + \sum_1^\infty C_n r^{-n} T_n.$$

Applying the usual boundary conditions

$$\phi_i = \phi_o \quad \text{on } r = a,$$

$$k(\partial\phi_i/\partial r) = (\partial\phi_o/\partial r) \quad \text{on } r = a,$$

we obtain

$$B_n = 2A_n/(1+k),$$

and

$$C_n = A_n a^{2n} (1-k)/(1+k), \quad n = 1, 2, \dots, p$$

$$= 0, \quad n > p.$$

Thus, apart from a possible additive constant, we get

$$\phi_i = 2\phi(r, \theta)/(1+k), \quad \phi_o = \phi(r, \theta) + \{(1-k)/(1+k)\}\phi(a^2/r, \theta)$$

and the above results for  $\Omega_i$  and  $\Omega_o$  follow.

2. In a similar manner if the field  $f(z)$  in air is caused only by singularities (charges, dipoles, etc.) inside the cylinder  $r = a$ , and if the cylinder is surrounded by homogeneous dielectric material of specific inductive capacity  $k$ , filling all the space for  $r \geq a$ , then the fields inside and outside  $r = a$  are respectively

$$\Omega_i = f(z) + \{(1-k)/(1+k)\}\bar{f}(a^2/z) \quad \text{and} \quad \Omega_o = 2f(z)/(1+k),$$

provided  $f(z)$  has no singularity at infinity.

3. For a logarithmic branch-point of  $f(z)$  at infinity, if we may expand the original potential in the form

$$\phi = A_0 \log r + \sum_1^p A_n r^{-n} T_n$$

on or near  $r = a$ , then the complex potentials become

$$\Omega_i = f(z) + \{(1-k)/(1+k)\}\bar{f}(a^2/z) + A_0 \log z$$

and

$$\Omega_o = 2f(z)/(1+k) + A_0 \{(1-k)/k\} \log z.$$

For an electromagnetic field the potential need not be single-valued, and so extra terms would have to be added to  $\Omega_i, \Omega_o$ .

G. POWER.

## REVIEWS.

**Logarithmetica Britannica, Part II.** Numbers 20,000 to 30,000, together with General Introduction. By A. J. THOMPSON. Pp. 106, cv, iii. 45s. 1952. Tracts for Computers, No. XXII. (Cambridge University Press)

This is the ninth and final part to be published of the *magnum opus* begun by Dr. Thompson in 1922 to commemorate the tercentenary of the publication in 1624 of the *Arithmetica Logarithmica* of Henry Briggs. (Readers of this review may like to be reminded that the title-page of this great work was reproduced as a frontispiece to the December, 1952, number of the *Gazette*.) The other eight parts were published at various dates between 1924, the tercentenary year, and 1937. The tables of the present part were sent to press in 1939; events since then have both delayed publication and allowed the author to write a fuller Introduction than he might otherwise have done. The now completed work is also to be issued in two cloth-bound volumes, the first containing the Introduction and the logarithms of numbers from 10,000 to 50,000, and the second containing the logarithms of numbers from 50,000 to 100,000. The whole publication has been sponsored by the *Tracts for Computers* Series edited at University College, London, first by the late Karl Pearson and now by Prof. E. S. Pearson. All numerical mathematicians will echo the author's appreciation of this timely, constant and effective patronage.

The main table in the present part is in the form made familiar by earlier parts. It gives in 100 pages the 20-decimal common logarithms of the integers from 20,000 to 30,000 inclusive, with second and fourth differences. All logarithms and differences which end in 5, 50, 500, . . . are followed by either a plus or a minus sign, in order that the table may be rounded to any smaller number of decimals without recomputation to decide doubtful roundings. This feature might well be more commonly adopted by makers of fundamental tables.

The complete work thus contains 20-decimal logarithms of the integers from 10,000 to 100,000. How much editorial labour during three centuries has been devoted to the logarithms of these same integers to fewer decimals! Errors in source tables such as those of Briggs, Vlacq and Vega have been found by one editor, overlooked by another, recognized again by a third, and so on. Dr. Thompson has now provided a source which we may reasonably hope to be definitive. The author himself appears to hesitate to claim complete freedom from error (compare his statements on pages xii and lxiv), but it is certain that the accuracy of the work is very great. This follows from the checks applied not only by the author, but also (in respect of the eight parts previously published) by the Mathematical Tables Project, now incorporated in the National Applied Mathematics Laboratories of the National Bureau of Standards, in the United States. It is significant that these checks revealed no error.

From a practical point of view, the tabulation of common logarithms of integers up to 100,000 is now finished. There would be little point in extending the logarithms of all these integers to more than Thompson's twenty decimals. On the rare occasions when logarithms to more than twenty decimals are required, the computer normally has recourse to special methods and tables, often quite compact, which are available in some variety.

Is there anything left which is worth doing? Broadly speaking, the answer appears to be: computation, no; publication, yes. When one contemplates the whole field of common logarithmic tabulation, one feels that publication has been excessive in some parts and deficient in others; it is impossible to be content with the present position in relation to the integers greater than

100,000. Apart from more extensive logarithms of the integers from 100,000 to 102,000, which have been published, the palm in respect of publication is held by the admirable 8-decimal table up to 200,000 published by Bauschinger and Peters in 1910. Yet it is well known that Edward Sang (1805-1890), of Edinburgh, computed logarithms up to 370,000 to 15 decimals, with errors of a few units in the last place, while logarithms of integers up to limits substantially beyond 100,000 have been calculated to 12 or more decimals on at least three other occasions.

All such calculations have remained in manuscript. Publication of extensive tables has always been difficult to arrange, and as the use of logarithms in numerical computation has declined, probably not much can now be expected. But several well-known logarithmic tables to seven or eight decimals give logarithms directly (i.e. without interpolation) up to at least 108,000, and there is no published fundamental table to which one may appeal to settle doubtful roundings. A table of Thompsonian accuracy, giving 12-decimal logarithms of the integers from 100,000 to 110,000, is the very least that should be salvaged, by competent checking and editing, from so much laborious computation. Such a table could be published in less space than the hundred pages occupied by each instalment of the main table in the *Logarithmetica Britannica*. The reviewer would prefer that at least twice this modest minimum amount should be rescued.

The final part of the *Log. Brit.* is a "double number" in both content and price. Besides the usual hundred pages of main table, it contains over a hundred pages of further material, including much of great interest. After the foreword and prefaces there is a 54-page Introduction. This deals fully with methods of interpolation and describes the processes by which the table was constructed. Chief emphasis is laid upon interpolation either by the method of factors (radix method) or by Everett's formula in terms of even differences, and these two methods are fully illustrated by worked-out examples of both direct and inverse interpolation; a radix table and an antilogarithmic table are included among the tables at the end.

The section on construction will especially interest numerical analysts. A large and important part of the main table was computed by subtabulation to hundredths of a basic table, to about 26 decimals, of the logarithms of all integers from 500 to 1000. This was prepared in the first instance from the 61-decimal logarithms of all integers up to 101 and all primes up to 1097 published in 1717 by Abraham Sharp in his *Geometry Improv'd*; but Sharp's deservedly famous table was later recomputed in its entirety by Thompson and was found to be free from error in the first 57 decimals (after making a correction given in Sharp's list of errata.) Some of Thompson's auxiliary tables relate to subtabulation in general, and are not restricted to the logarithmic case. There are valuable discussions and suggestions about table-making problems.

Two facts may be mentioned in illustration of the author's keenness and resourcefulness in utilizing modern technical aids. The first is that, in pursuit of efficiency in calculation, he devised for himself, and had constructed, an "integrating and differencing machine" which is illustrated by a photograph and described in the text. The second is that, in order to reduce the cost of printing the tables, and so make publication financially possible, he purchased a "Monotype" keyboard and its ancillary apparatus, and himself punched the holes in the ribbon of specially prepared paper which controls the casting of the type; the subsequent operations were performed by the Cambridge University Press.

Appendix (i) contains, in ten pages, a very readable translation into English, by Mr. John Theodore Foxell, of the memoir on the life and work of Henry

Briggs originally published in Latin by Thomas Smith in his *Vitae quorundam eruditissimorum et illustrium virorum* of 1707. One enjoys the studied classicism and charming hyperbole now chiefly reserved for the use of Public Orators. Included in the translation are the title-page and Henry Gellibrand's editorial preface from the great *Trigonometria Britannica*, published in 1633, two years after the death of Briggs:

"To students of the mathematics Henry Gellibrand giveth greeting.

"How untimely always and how disastrous to the republic of letters is the death of those who are engaged on immortal work, I would that we had learned otherwise than through the death of our most learned Briggs, certainly the marvel of this age of mathematics. Then indeed neither would posterity have been bereaved of this light, nor would the burden of so great a task have been laid upon my weakling shoulders, or so great an unevenness have been detected in the weaving of this fabric. . . ."

Appendix (ii) consists of a six-page list of errors in the *Arithmetica Logarithmica* of Briggs. This is followed by the Factorizing Table and the Antilogarithms, both already mentioned. Then come a page of short tables and constants, and a 21-decimal table of logarithms of integers up to 1000. At the end are title-page and table of contents of Volume II, for the convenience of anyone who wishes to bind the nine parts in two volumes.

Dr. Thompson is to be warmly congratulated on the completion of his lengthy task. The whole work will long remain a landmark, perhaps the last of the great landmarks, in the history of the development of logarithmic tables. In addition, Part II will appeal to those who are interested in the general subject of numerical analysis or in the history of mathematics.

A. FLETCHER.

**Mathematical Models.** By H. M. CUNDY and A. P. ROLLETT. Pp. 240. 2ls. 1952. (Oxford University Press).

This is a first class book, beautifully produced, full of excellent matter—as indeed one would expect from these two authors with their wide experience—and not confined to the nets of the lesser dodecahedron but having a wide range even up to the elementary constituents of a differential analyser. It is virtually a handbook both for the beginner and for the more experienced model-maker, for within the various chapters are collected in a clear and precise fashion all the necessary details and instructions for making the models.

The first chapter deals with the mechanics of model-making of all types and discusses the relative merits of the different media and fixatives, and the second with a large number of models in plane geometry covering such diverse topics as curve-stitching and the golden section. Polyhedra are dealt with fully in the third chapter, and all the models one could wish for in solid geometry from wire models to sphere-packs in the next. Finally, mechanical models, including linkages, and machines for drawing curves and solving equations complete the contents; an excellent index is preceded by a short bibliography.

The book is printed on good quality paper, has clear type and bold headings, and, most valuable of all, is profusely illustrated with beautifully drawn diagrams and illustrations, supplemented by four pages of plates.

This volume should find a place in every mathematical library and will be of inestimable value to a teacher in any type of school from secondary modern to technical, grammar or university level. Mathematics teachers have been waiting a long time for just such a book as this, and the delay has been more than justified by this publication.

B. J. F. D.

**Cartesian and projective geometry.** By R. WALKER. Pp. 320. 21s. 1953. (Arnold)

*Cartesian and projective geometry* is not, as the title suggests, exclusively concerned with cartesian co-ordinates, but contains an account of the method of generalized homogeneous co-ordinates applied to curves which are usually of degree 2 or less. The approach to Projective Geometry is non-metrical, by (1, 1) correspondence of parameters, and metrical results appear as special cases. Although the author claims that no attempt was made to keep analytical and pure methods separate, the former preponderates and it is only in the last three (of twelve) chapters that we see the incisive Projective methods at work. The student is, very sensibly, advised not to leave these last chapters until he has absorbed all of the preceding work, and these can be studied as soon as the ideas of related ranges and cross-ratios have been established. A chapter on Reciprocation concludes the work, except for an appendix wherein the hazardous words "Project A and B into I and J" are vested with a comfortable reality. There is a wide range of topics and few of the familiar properties of conics—general and particular—are omitted. The examples are well chosen, many of them from the Cambridge Scholarship questions, and the solutions given in the text are models. The printing is excellent and misprints are not ubiquitous. (I have discovered none.)

It is essentially a book for the Scholarship candidate. He, if equipped with but part of the bookwork and a few of the methods offered him, need have no fear of the scholarship paper labelled "Geometry". As an undergraduate achieving a friendly familiarity with geometry, he will find much useful material (and he will find it quickly, too, for the table of contents is commendably comprehensive and there is a generous index besides) and stimulating problems whose solutions he can compare with the elegant ones given in the text; and if his interest should persist until middle age, he will find much in this book to awaken old enthusiasms. I can vouch for one such case.

W. J. HODGETTS.

**Calculus and analytic geometry.** By G. B. THOMAS. 2nd edition. Pp. xi, 731. \$8.50. 1953. (Addison-Wesley Press, Cambridge, Mass.)

The first edition, reviewed in the *Gazette*, Vol. XXXVII, p. 160, was reproduced from typescript, so as to allow for revision before the definitive setting in type. The printing in the second edition is clear, the diagrams are well drawn, and the whole book is easy in style, with plenty of worked and unworked examples.

Allowing for the changes in outlook during the past 40 years, Thomas' book is perhaps most aptly equated to the old-established English favourite, G. W. Caunt's *Infinitesimal Calculus*. Caunt gives more applications to applied mathematics, Thomas takes the pure mathematics further by including complex variable, multiple integrals and coordinate geometry of three dimensions. But the general scope and aim do not differ greatly in the two books, nor is Thomas' style markedly inferior to the simple and lucid ease of Caunt's. For the student with a reasonable grounding in algebra, geometry and trigonometry and an ultimate objective in physics or engineering, this volume would be a most helpful text. Integration is brought in at a reasonably early stage, and a chapter on differential equations, chiefly linear with constant coefficients, is included.

T. A. A. B.

**On the metamathematics of algebra.** By A. ROBINSON. Pp. ix, 195. Fl. 18. 1951. (North-Holland Publishing Co., Amsterdam)

The application of mathematical methods in the field of logic created the science of symbolic logic; in the "metamathematics of algebra" Professor

Robinson starts to repay the debt which logic owes to mathematics, and uses the methods of symbolic logic to generalise the concepts of modern algebra. Despite the title, which could hardly be more misleading—though its motivation is clear enough—the *Metamathematics of algebra* is a book for algebraists not logicians; its claim to a place in the new series of studies in logic and the foundations of mathematics is very slender. Only the first fifth of the book, which is devoted to an extension of Gödel's completeness theorem to non-denumerable systems of statements, has any bearing on the foundations of mathematics, and the remaining four-fifths may be read without reference to this first part which could with advantage have been omitted. The author's attitude to foundation problems, in fact, is simply to ignore them.

Whatever the proper description of its field of study may be, there is no doubt of the quality of this book. The volume is brilliantly conceived and its appearance marks another milestone on the road of mathematical discovery.

One of the primary objects of the book is to set such fundamental concepts of algebra as *algebraic number*, *polynomial*, *ring* and *ideal* free from the arithmetical operations in which they were conceived. The generalised polynomials, for instance, are a sub-class of a set of relations called *prepolynomials*. A prepolynomial is a relation  $R(x_1, x_2, \dots, x_n, y)$  which satisfies the conditions

$$(x_1)(x_2) \dots (x_n)(\exists y)R(x_1, x_2, \dots, x_n, y) \\ (x_1) \dots (x_n)(y)(z)\{R(x_1, \dots, x_n, y) \& R(x_1, \dots, x_n, z)\} \rightarrow E(y, z);$$

here " $( )$ ", " $\exists$ " are the universal and existential quantifiers, " $\&$ ", " $\rightarrow$ " denote conjunction and implication, and the function  $E$  expresses an equality relation. These conditions simply affirm the existence of a unique  $y$  having the relation  $R$  to the ordered set of  $x$ 's. Polynomials (in  $x_1, x_2, \dots, x_n$ ) are the prepolygons which have the particular form

$$(\exists z_1)(\exists z_2) \dots (\exists z_m)Q(z_1, z_2, \dots, z_m, x_1, x_2, \dots, x_n, y)$$

where  $Q$  itself contains no quantifiers and is formed by the conjunction of two term relations. An example given of a generalised polynomial is

$$E(x_1, x_1) \& E(x_2, x_2) \& \dots \& E(x_{i-1}, x_{i-1}) \& E(x_i, y) \& E(x_{i+1}, x_{i+1}) \& \dots \& E(x_n, x_n)$$

The generalisation of an ideal is more simply expressed. If  $K$  and  $J_0$  are two sets of statements in a language  $L$  then a subset  $J$  of  $J_0$  is called an ideal in  $J_0$  over  $K$  if all the statements of  $J_0$  which can be deduced from the totality of statements in  $J$  and  $K$  are included in  $J$ . A number of fundamental theorems on generalised ideals are obtained, including a necessary and sufficient condition for an ideal to be irreducible and a proof that every ideal is the meet of a finite number of irreducible ideals. The generalisation is shown to preserve the idea of a basis but the concepts of prime ideals and primary ideals are not carried forward. If the domain  $J_0$  necessarily contains the disjunction of any two of its statements, the ideals of the domain are called disjunctive ideals. Although they have no direct counterpart in classical ideal theory the introduction of disjunctive ideals is shown to lead to a variety of interesting results, even apart from their bearing on the theory of polynomial ideals.

Another direction in which the "metamathematics of algebra" achieves a generalisation of the methods of modern algebra is in the derivation of non-trivial mathematical theorems from general propositions in symbolic logic. The following is an outstanding example of the application of this technique: if  $S$  is a monotonic increasing sequence of statements, that is, a sequence in which each statement is implied by (but is not equivalent to) any of its successors (in some formal language  $L$ ) then there is no single statement in  $L$  which implies all the statements in  $S$  and is in turn implied by them. For if

$Y$  is such a statement then it is derivable from some subset of  $S$ , say  $X(n_1), X(n_2), \dots, X(n_k)$  (in which we may suppose the  $n$ 's arranged in increasing order). Since  $S$  is an increasing sequence, each  $X(n_r)$  can be deduced from  $X(n_k)$ ,  $1 \leq r \leq k$ , and so  $Y$  is derivable from  $X(n_k)$ ; but all the statements of  $S$  are derivable from  $Y$ , and therefore in particular  $X(n_k + 1)$  is so derivable, so that  $X(n_k + 1)$  is derivable from  $X(n_k)$  which contradicts the assumption that  $X(n_k)$  is not equivalent to any of its successors. From this simple argument, by a suitable choice of the sequence  $S$ , Robinson deduces amongst other results:

1. Any theorem of the restricted predicate calculus which is true for all non-archimedean ordered fields is true for all ordered fields.
2. Any theorem of the restricted predicate calculus which is true for the field of all algebraic numbers is true for any other algebraically closed field of characteristic 0.
3. If a set of polynomial equations with integral coefficients has no solution in any extension of the field of rational numbers then it has no solution in any field of characteristic  $p > p_0$ , where the coefficients are taken modulo  $p$  and  $p_0$  is a constant depending on the set of polynomials.

How fertile this union of algebra and symbolic logic promises to be!

R. L. GOODSTEIN.

**An essay in modal logic.** By G. H. VON WRIGHT. Pp. vi, 90. Fl. 9. 1951. (North-Holland Publishing Co., Amsterdam)

The formal axiomatic method which has been so successfully employed in logic and mathematics is applied by Professor Wright in this essay to a variety of concepts of seemingly very different kinds, like possibility, obligation and permission. The formal systems constructed to express these concepts are named modal logics; these systems are all of a very simple character and each consists in effect of the adjunction of an undefined element to the familiar calculus of propositions to express one of the modal concepts. For instance, the axioms for *possibility*, expressed by means of the operator  $M$ , are, in addition to a set of axioms for the propositional calculus, the following four:

- |                           |  |
|---------------------------|--|
| 1. $a \rightarrow Ma$     | 2. $M(avb) \leftrightarrow Ma \vee Mb$ . |
| 3. $MMa \rightarrow Ma$ . | 4. $M \sim Ma \rightarrow \sim Ma$ .     |

The first says that a true proposition is possible, the second that the disjunction " $a$  or  $b$ " is possible if and only if  $a$  is possible or  $b$  is possible. The third axiom resolves to make no distinction between possibility and the possibility of possibility, while the fourth says that a proposition is impossible if its impossibility is possible.

It is shown that the modal logic for possibility is closely akin to the logic of quantifiers. Professor Wright points out that this analogy is to be expected since (in popular phraseology) the possible is that which is true under some circumstances, the impossible that which is true under no circumstance and the necessary is that which is true under all circumstances.

The idea of a disjunctive normal form, which is important in the propositional calculus, is shown to have an analogue in modal logic and by its means the decision problem for modal logics, that is, the determination of a finite procedure for testing the provable formulae of the logics, is solved completely.

The essay is written for beginners in a simple pedagogic style with ample repetition and lots of capital letters for principles, and presupposes no technical knowledge beyond a familiarity with a formal treatment of the logical connectives.

R. L. GOODSTEIN.

**Elementary Calculus.** By A. KEITH and W. J. DONALDSON. Pp. 425, lxxii. 16s. 6d. 1952. (Robert Gibson and Sons)

This book has something in common with the Capriol Suite: old tunes in a modern setting. It might have been written by taking a much-admired book published thirty or forty years ago, and presenting its material afresh with all the teaching-skill gained in the intervening years. The result is by no means displeasing, and often refreshing.

Part I, which may be obtained separately, deals carefully with the differentiation and integration of  $x^n$  and  $(ax+b)^n$ , and applies this technique to a wide range of problems. Leibnitz' notation is used, with  $dy/dx$  as a single symbol, the work is broken up into short sections each followed by examples, and it is admirably clear. Part II is more severe. Each chapter is long, and examples come only at the end of it. That on differential equations, for instance, consists of 23 pages in which all the methods of solution are explained, followed by 100 examples for the student to work. The teacher will know how to handle this, but the private student may find the going hard. The ground covered meets all school requirements except perhaps those of mathematics specialists, and should also be adequate for scientists and engineers in their first year at the university.

It is in the use of infinite series in Part II that the book looks backwards least happily. The exponential function is introduced by means of the differential equation  $dy/dx=y$ . The authors then state that the solution is the infinite series  $y = 1 + x + x^2/2! + \dots$ , and verify this by differentiating term by term, with, of course, due warning that infinite series will only stand such treatment in suitable circumstances. This approach leads to quick results for the exponential and logarithmic functions, but may retard a correct understanding of infinite series. In the chapter on expansions, the infinite series take first place, and the use of the first few terms as an approximation to the value of the function is made secondary; the Calculus Report would reverse the emphasis.

Throughout the book the print and diagrams are large, clear, and easy to read, and the eye is not distracted by frequent changes in the size and kind of type. It is perhaps in order to secure these advantages and yet keep the size and price of the book reasonable that the authors keep closely to the highway of calculus and rarely wander into the bypaths. There is much to be said for this, and teachers who want a straightforward book would do well to examine this one.

A. H. G. P.

**Revision Course in General Mathematics.** By C. V. DURELL. Pp. vi, 153. With answers, 6s.; without answers, 5s. 6d. 1952. (G. Bell and Sons)

This is for the school year in which mathematics is taken at Ordinary level. It contains material for two terms' work only, so that the few remaining weeks may be spent on actual examination papers. There are five sections, dealing respectively with Arithmetic, Algebra, Geometry, Trigonometry, and Revision Papers, easy and hard, followed by almost all the four-figure tables which could be desired, except reciprocals. The first four sections consist of exercises, each preceded by a list of the facts or formulae relevant to it, or by a worked example. Everything the pupil needs to know is to be found in the lists, except proofs of theorems and formulae and ways of doing geometrical constructions. Teachers who introduce this book will probably want their pupils to retain their normal text-books as well, but the new book should supply a stimulus in what might otherwise be an uninspiring year. It is of the high quality which teachers have come to expect from its author and publishers, and is recommended.

A. H. G. P.

**Finite Matrices.** By W. L. FERRAR. Pp. vi, 182. 17s. 6d. 1951. (Oxford University Press)

The author continues his series of text-books on algebraic topics which now lead the reader from advanced school work to post-graduate level and almost to the threshold of original research. The volume under review contains selected material from the more advanced theory of matrices: equivalence of matrices in the complex field and of  $\lambda$ -matrices in the polynomial ring of one indeterminate, collineations including elementary divisors, congruence with orthogonal (unitary) equivalence and quadratic (Hermitian) forms, infinite series and functions of matrices, and matrix equations. Most of this is at present not usually included in an Honours course at our universities, although it is to be hoped that before long the standard theories of matrix equivalence, collineation and congruence, will be considered an indispensable part of the equipment of every Honours graduate.

The book makes easy reading as a result of the author's deliberate policy. His standpoint is "classical" rather than "modern". His treatment is direct and not axiomatic. His matrices have complex or real coefficients throughout. He does not strive for the greatest generality of his results nor for the shortest proofs. His guiding principle is to make things easy for the reader and to avoid subtle arguments which he believes are difficult to follow. To give an example: the chapter on collineations, the Jordan canonical form, and elementary divisors occupies nearly forty pages whereas van der Waerden covers about twice the material in half the space. Probably it must remain a matter of personal taste and of mathematical upbringing whether one prefers an appeal to a long series of (admittedly simple) calculations to concentrated mathematical reasoning.

The book is not quite free from minor blemishes. Among those noted by the reviewer are the following: 1. The symbol  $A_{ij}$  associated with a matrix  $A$  occurs in two different meanings on pp. 4 and 16; the property described as "reflexive" on p. 135 should be "symmetric"; the use of the term "nilpotent" for a matrix whose square (not an arbitrary power) is zero is very unusual. 2. The reviewer cannot see the point of introducing the concept of a number field on p. 15. If it is not assumed to be algebraically closed then the characteristic roots which are required throughout need not exist and if they do, need not lie in the field. The process of "adjunction" which is referred to on p. 54 remains entirely unexplained. It would seem that one should either confine the discussion consistently to the field of complex numbers or develop the general field theory properly. Incidentally the "short proof" on p. 80 is, of course, based on a vicious circle because it assumes an independent knowledge of the existence and main properties of the elementary divisors. 3. For the simultaneous reduction of two quadratic forms to sums of squares the usual sufficient condition (one form positive-definite) is given. This is good enough for the simplest geometrical applications, but in an algebraic context it seems desirable to give necessary and sufficient conditions.

The chapter on infinite series and functions of matrices is very interesting. It contains material which has never been in a text-book before, much of it the result of the author's own research.

K. A. H.

**Introduction to Modern Algebra and Matrix Theory.** By O. SCHREIER and E. SPERNER. Translated by M. Davis and M. Hausner. Pp. viii, 378. \$4.95. (Chelsea Co., New York)

Otto Schreier's *Einführung in die Algebra und Analytische Geometrie* (edited by E. Sperner) has for many years been a favourite text-book for first year undergraduates in German universities. One of its unique features is the simultaneous treatment of fundamental concepts in algebra and in affine and

projective geometry. The present skilful English translation has incorporated the "Vorlesungen über Matrizen" (as in the German second edition), but has omitted the long final chapter on projective geometry in  $n$  dimensions. The result is that the accent is much more on the algebraic developments than in the original and that the geometry serves more as illustration than as motivation. This is also expressed in the changed title from which the reference to geometry has disappeared. Nevertheless it is a little misleading to call the book an introduction to "modern" algebra because it contains rather little group theory, no treatment of the concepts of isomorphism and homomorphism, no theory of rings and ideals, and only those parts of field theory which are required in the applications. "Linear" algebra would be more appropriate.

Apart from this change of emphasis the book has preserved its carefully didactic character. The thorough explanations of concepts, the meticulous statements of assumptions, the elegance of treatment (which is not always brief, but always lucid) deserve the highest praise. The first chapter introduces the affine space on  $n$  dimensions and gives a complete discussion of linear equations (free from determinants). Determinants are introduced (by way of volumes of parallelepipeds) and developed in the second chapter which deals with Euclidean space. It also treats orthonormal bases and rigid motions in two- and three-dimensional Euclidean space. The third chapter on field theory culminates in a (classical) proof of the so-called Fundamental Theorem of Algebra. The elements of group theory are contained in the short fourth chapter which leads as far as the basis theorem for Abelian groups with finite numbers of generators. The long fifth chapter is devoted to linear transformations and matrix theory. It gives a very clear account of the classical and the Jordan canonical form together with the theory of elementary divisors and of orthogonal and unitary transformations and the principal axes theorem (for a single matrix, not for pairs).

The book can be thoroughly recommended both for private study and as a text-book for an enterprising Honours course combining algebra with geometry. It will have to be supplemented in both directions for part of the second and third year work—but as an introduction it can hardly be surpassed.

K. A. H.

**Calcul Vectoriel. Tome 1. Algèbre. Algèbre Linéaire, Applications.** By LUCIEN CHATTELUN. Pp. viii, 605. 5,000fr. 1952. (Gauthier-Villars, Paris)

When reading a modern mathematical publication, how often do we regret the leisurely style, the patient exposition, the contemplative reflection and recapitulation which was current in mathematical papers at the turn of the century. The prohibitive costs of printing and the scarcity of paper, as well as the present day tendency to mathematical abstraction and brevity of statement have combined to produce a mathematical style which not infrequently can be comprehended only with the greatest difficulty. In comparison with present day works the writings of Frobenius, Scheffers, Salmon and their contemporaries are monuments of lucidity. Volume I of Lucien Chattelet's *Calcul Vectoriel* may be said to be written in the old style, in as much that no considerations of space have prevented the author from saying all that he wanted to say and, indeed, almost all that can be said on Vector Algebra of three dimensions and its applications to trigonometry, Euclidean geometry and analytical geometry. Clarity of statement, abundance of illustrations and rigour of argument make this book a pleasure to peruse, if one has the leisure and inclination to read it. It contains over 600 pages roughly Royal 8vo. in size. The first four chapters occupying 100 pages deal with vector addition, multiplication of a vector by a scalar and all that arises from these concepts. Three chapters on scalar and vector multiplication with their applications

occupy the next 180 pages, while the remaining two chapters deal, one with the metrical group of 3 dimensional geometry and its isomorphism with the orthogonal group, the other with systems of localised vectors (*glisseurs*). Four extensive notes complete the book. The first of these is concerned with the theory of vectors in Euclidean space of  $n$  dimensions. The second deals with the vector product of  $n-1$  vectors in  $n$ -dimensional space, developing simultaneously the relevant parts of determinantal theory. The third note is on quaternions which are introduced as operators acting on a three dimensional vector. The last note introduces linear operators and linear vector spaces thus bringing the student up to the very threshold of present day mathematical thought. According to the preface, this work attempts to give a "formal exposition of vector calculus, rigorous and complete, with concrete applications as numerous and varied as possible taken from the different branches of mathematics".

Arguments which make explicit appeal to geometry or to a system of Cartesian coordinates have been deliberately kept in the background in the development of the theory in order to give the book a logical unity. Such a procedure certainly emphasises the great flexibility of vector algebra, but, as the author points out, the vector treatment does not always afford the shortest or the most elegant demonstration of a theorem or property. Thus almost three pages are devoted to solving the equations  $a_1x + b_1y - c_1 = a_2x + b_2y - c_2 = 0$  by vector methods, while triple scalar products precede the definition of a determinant and the derivation of its properties.

Although a familiarity with only some of the more elementary mathematical theories is required of the reader, the book is addressed to those, both professional and amateur, who possess some mathematical maturity. The author also suggests that it is suitable for students of mathematics and physics and, in particular, for candidates for the *Aggregation* degree. In the reviewer's opinion, however, it is unlikely to suit British physicists since none of the many illustrations are taken from the realm of mathematical physics. Much space, also, is devoted to niceties which might seem to be irrelevancies to the average physicist. The notation which is employed will certainly be found cumbersome

and confusing by the reader of English and German text books. Thus  $\vec{a}$  denotes a vector,  $|\vec{a}|$  its length,  $\vec{a} \times \vec{b}$  a scalar product,  $A \uparrow B$  a localised vector. The emphasis on Grassman forms of the first and second species makes the development of the subject more sophisticated than is usual. Any mathematician who can spare five guineas and a good deal of time will find this an interesting and readable book.

D. E. RUTHERFORD

**Analytic Geometry.** By JOHN W. CELL. 2nd Edition. Pp. xii, 326. 30s. 1951. (John Wiley, New York; Chapman & Hall)

This work, after discussing preliminary concepts in plane analytic geometry, develops in considerable detail methods of curve sketching and also the converse business of locus derivation, i.e. finding the equation of a curve which has given geometrical properties.

Chapters entitled Conics, Transcendental Curves, Polar Coordinates and Parametric Equations follow and a chapter on Empirical Equations closes this part of the book. Two final lengthy chapters furnish an introduction to planes, lines and surfaces in solid analytic geometry.

The author presents all the foregoing material in an easy, almost leisurely fashion and throughout the book lays great stress on practical graphical exercises. Examples are plentiful, the diagrams clear.

An appendix includes lists of formulae, definitions and 4 figure tables of common and Napierian logarithms, natural sines, cosines, tangents and cotangents. One page tabulates  $\exp(u)$  and  $\exp(-u)$ .

Whilst it is difficult to bring to mind any mathematics course in an English school or college for which this would prove a suitable text-book, certainly no such establishment should neglect to add a copy to the library shelves; it should interest both students and teachers. A private student especially might find its gentle approach helpful for supplementary reading.

J. K.

**Solid Geometry.** By WALTER W. HART and VERYL SCHULT. Pp. x, 198. 12s. 6d. 1952. (D. C. Heath and Co., Boston; George G. Harrap and Co.)

This book has the strangeness combined with freshness of approach that we associate with American text-books, but which make it difficult to incorporate in any English scheme of teaching. The preface states that an attempt has been made to reduce the quantity of formal demonstration in the course, and to lay emphasis on mensuration, but the book remains, to English eyes, excessively formal. Proofs are often merely sketched in outline, but the development is along axiomatic lines and would be difficult to English children who have long ago dispensed with formal "proofs" of congruence and similarity theorems. The fundamental difficulty to the reviewer is to decide at what stage this book could be used in an English Mathematics course. Adequate facility in (Stage C) formal plane geometry would be needed, but calculus is expressly excluded as "higher mathematics", and the mensuration results obtained are for the most part elementary. The "Optional Topics" give glimpses over the wall, into such fascinating territory as Conic Sections, Areas of Curved Surfaces, Spherical Triangles, Map Projections, but they mostly leave off just where an English sixteen-year-old would begin to find them interesting.

The book is well illustrated with examples of three-dimensional objects in real life—cones for example appear in mountains and table-lamps—and the plentiful exercises are commendably concrete and practical. I venture to think however that experience of candles and of children with candy does not justify the following illustration of the approach to a limit (Optional Topic E): "The length of a burning candle is a decreasing variable that approaches zero. If the candle is allowed to burn out, its length finally becomes zero. If a child, having some candy, eats half of it; next eats half of the remainder; then continues, always eating half of the remainder; the remainder is a decreasing variable that approaches but does not become zero, because the child each time eats only half of the remainder." I commend to harassed parents this solution of the age-old problem of how to eat your candy and have it.

Volumes are approached as usual by assuming Cavalieri's Theorem that two solids are equal in volume if they can be placed with a pair of opposite faces in the same two parallel planes, and if all sections by planes parallel to these planes are equal in area. The sphere is measured by a method closely akin to that used by Archimedes (though this interesting fact is not stated), and the formulae for spherical segments, frusta, etc., are all derived from a basic result that the volume of a *prismatoid* (polyhedron with all its vertices in one or other of two parallel planes) is one-sixth the height multiplied by the sum of the areas of the parallel bases and four times the area of the section midway between them. The method is a powerful one, and could well have been developed more systematically. This and the section on spherical geometry are to the reviewer the most interesting parts of the book.

There are copious "review exercises", mainly of the "quiz" type, which go far to ensure that the pupil will know the contents of the book. How far such a course will lead him to think for himself in three dimensions is rather more problematical.

H. MARTYN CUNDY

**Partial Differentiation.** By R. P. GILLESPIE. Pp. 107. 6s. 1951. University Mathematical Texts (Oliver & Boyd Ltd.)

This addition to the University Mathematical Texts covers the basic theory of partial differentiation. Without assuming any previous knowledge of theory of functions of several variables, it goes well beyond the needs of the General degree student, and while it deals adequately with most topics that a Special student would need for examinations, it does rather limit itself to those ends. The usual geometrical applications to curvature, multiple points and envelopes, are supplemented by a treatment of curvilinear coordinates, the elements of conformal transformation, and some vector analysis. The treatment of differentials is on conventional lines. The first differential of a function of two variables is clearly defined as a function of four variables  $x, y$  and  $dx, dy$ , but the higher differentials are not so clear. The beginner will doubtless be puzzled why  $d^2u$  is homogeneous in  $dx$  and  $dy$  while  $d^2f$  is not so when  $x$  and  $y$  are functions of  $u$  and  $v$ . Maxima and minima are dealt with fully, though it is odd to meet the statement that if a function  $f(x)$  has a turning value and all derivatives at  $a$  then  $f^{(n)}(a)$  must differ from zero for some  $n$ . Lagrange's method of undetermined multipliers receives special attention, and a quite extended discussion is given of sufficiency conditions for a turning value and for its discrimination. There is a plentiful supply of examples on transformation of variables.

H. K.

**Plane Trigonometry.** By J. TOPPING. Pp. viii, 302. 10s. 1952. (Longmans Green)

This is a Sixth Form textbook, aimed at meeting the needs of all but mathematical specialists in the Grammar Schools. The book starts with a rapid revision of elementary work on the six ratios, their general values, graphs, simple relationships between the six ratios, and simple equations. A clear diagrammatic approach to the question of angles with a given trigonometrical ratio is lacking, and would have helped the reader's understanding in the work on simple trigonometric equations.

The sine and cosine rules, areas and projections, are very well covered in the second chapter, where a treatment of sector areas, involving a discussion of  $(\sin \theta)/\theta$  when  $\theta$  is small, is adequately carried out.

There follows a chapter headed "Vectors", where simple applications of the idea of a vector sum are made, and the expression of  $\cos A + \cos B$  as a product is done using vectors. The transformation

$$a \sin \theta + b \cos \theta = r \sin(\theta + \alpha)$$

is also carried out, and leads to the solution of various trigonometrical equations. No mention, however, is made of the product of two vectors. A further chapter is devoted to kindred work on  $\sin(A \pm B)$  and further identities.

In chapter V a systematic approach to the solution of triangles is made, with full working instructions. The treatment of the formulae for  $\tan \frac{1}{2}A$ ,  $\tan \frac{1}{2}B$ ,  $\tan \frac{1}{2}C$  leaves much to be desired, however. By using the form

$$\tan \frac{1}{2}A = r/(s-a), \quad \text{where } r = \sqrt{\{(s-a)(s-b)(s-c)/s\}},$$

the calculation of all three angles can be made to take very little longer than that of one angle and an effective check is obtained.

In Chapter VI (Small angles and approximations) much work is done on questions involving small errors, in which a calculus treatment would usually be preferred. There is an excellent set of exercises on the approximate solution of trigonometrical equations, but no general method (such as Newton's) is indicated.

After a set of 150 miscellaneous examples of Advanced Level standard, two chapters are devoted to properties of the triangle and quadrilateral, which one would suppose to be of little use to any but specialists in mathematics. The

book is completed with a brief treatment of inverse trigonometrical functions and a set of 50 harder miscellaneous examples.

The type is clear, perhaps being a little large for the size of page, thus leading to the result that many proofs occupy more than one page. There are few misprints—there is one bad one at the foot of p. 187—and explanations are clear and concise. On p. 6 the argument about negative values of the radius vector in polar coordinates is not very convincing, but this is a minor defect in an otherwise clear treatment of the subject.

A chapter or two on complex numbers, de Moivre's theorem, and vector products would have increased the scope of the book considerably. As it is the appeal of the book in grammar schools is bound to be rather limited.

F. J. TONGUE.

**Differential und Integralrechnung.** By W. MAÄK. Pp. 235. N.p. 1949. (Wolfenbütteler Verlagsanstalt)

This book gives a readable account of the elementary portions of analysis; it does not resemble an English book on calculus, in which we expect instruction on particular problems such as maxima and minima, curvature, areas and volumes, and methods of evaluating many types of integral. It begins with a description of the necessary fundamental properties of numbers. Then the first part of the book discusses the following topics: the idea of limit, convergence of sequences, continuity, differentiation, Rolle's and the mean value theorem, interpolation, elementary results on infinite series, the Riemann integral and the improper integrals derived from it.

The other part of the book introduces functions of two variables and develops on interesting lines the theory as far as Green's transformation and the formula for change of variable in a double integral. Two problems arise here. The first and more elementary one is that of establishing the formulae of the subject, for example, length of a curve, area of a region, Green's transformation, under no matter how restrictive conditions; the other is that of extending to wider conditions. To have included anything on the latter problem would have been to go outside the natural scope of the book. The author restricts himself to curves built up of a finite number of arcs with continuously turning tangents and to regions bounded by a finite number of such arcs. Even so, the statement and argument of the approximation to

$$\iint f(x, y) dx dy$$

by sums of the form

$$\sum f(x_i, y_j) \Delta x_i \Delta y_j$$

needs some modification because parallels to the axes may meet the boundary of the region in an infinite number of points.

The definition of differentials is unusual. Functions  $t=t(x)$  which are continuous and have continuous non-vanishing first order derivatives in an interval  $a \leq x \leq b$  are called parameters for the interval. If  $f(x)$  has a continuous derivative for  $a \leq x \leq b$ , the author defines its differential  $df$  to be the derivative of  $f$  with respect to an arbitrary parameter of the interval. Thus for  $f(x)=e^x$  and the interval  $-\frac{1}{2}\pi \leq x \leq \frac{1}{2}\pi$ , if we take  $x$  as the parameter we had  $df=e^x$ , but if we take  $t=\sin^{-1}x$  as parameter we have  $df=e^x \sqrt{1-x^2}$ .

For functions of two variables the author defines  $f(x, y)$  to be differentiable at a point  $p$  if  $f$  has continuous first order partial derivatives throughout some neighbourhood of  $p$ . He points out that this is not the usual definition but adopts it to simplify his discussion. As the differential of a function of one variable was defined in terms of an arbitrary parameter, so the definition of  $df(x, y)$  involves an arbitrary curve. If  $f(x, y)$  is differentiable in a domain  $D$ , we take an arbitrary smooth curve in  $D$  with equations  $x=x(s)$ ,  $y=y(s)$  in terms of its arc length  $s$ . Then  $df(x, y)$  is defined to be  $dF(s)$ , where  $F(s)=f(x(s), y(s))$ .

These ideas are used in the discussions of Jacobians and of curvilinear integrals, but the book is not extensive enough to enable a proper estimate to be made of their value. Grave doubts arise about the wisdom of introducing a symbol with such a variable meaning as the author's  $df$ .

The paper on which the book is printed is of poor quality and in places the reader has to supply portions, sometimes the whole, of a symbol. Other misprints are rare and I have noted only one: in line 24 of page 71,  $n$  should be replaced by  $n-1$ .

R. C.

**Grundzüge der Mengenlehre.** By F. HAUSDORFF. First (1914) edition, reprinted 1950. \$4.95. (Chelsea Co., New York)

The second edition of this book was published under the title *Mengenlehre* and omitted several topics which were in the 1914 edition. It was a matter for regret that Hausdorff did not write another book on those omitted topics. We therefore welcome the reprint of the original edition as it makes one of the classics of the theory of sets again available.

R. C.

**Semantics and the Philosophy of Language.** Edited by L. LINSKY. Pp. ix, 189. \$3.75. 1952. (University of Illinois Press, Urbana)

This is a photostatic reproduction of thirteen previously published essays on the meaning of meaning, with a brief but helpful explanatory introduction by the editor. All but one of the essays were written within the last decade, the single exception being Russell's account of the theory of descriptions which is taken from his *Introduction to mathematical philosophy*. One of the aims of this theory of descriptions is to find a sense for reference by description to non-existent objects. For instance, in denying that Pegasus exists we seem forced to find some object which "Pegasus" names, some *unactualised possible*, since if Pegasus denotes nothing what is that of which we are denying the existence. Russell solves this puzzle by analysing descriptions in terms of bound variables. Thus "the author of *Waverley* was a poet" is analysed as "there is an  $x$  such that  $x$  wrote *Waverley* and  $x$  was a poet, and if  $y$  wrote *Waverley* then  $y$  is the same as  $x$ ", and "there are no square circles" becomes "for all  $x$ ,  $x$  is a square and  $x$  is a circle are not both true". Instead of inventing an imaginary 'square circle' for its reference the description "square circle" is shown to be an eliminable linguistic form. In his paper "On what there is", Quine carries this analysis a step further and shows that names, too, are eliminable (by first translating them into descriptions), a situation which Quine sums up by saying that "to be is to be the value of a variable".

In a second paper, "Notes on existence and necessity", Quine discusses some apparent failures of the law of substitution of identicals. For instance, from the fact that Cicero and Tully are names of the same man, we obtain from the true sentence "'Cicero' contains six letters", by substituting 'Tully' for 'Cicero', the false sentence "'Tully' contains six letters". Quine seeks to resolve this paradox by distinguishing between a designative and a non-designative use of a word or phrase. Thus 'Cicero' and 'Tully' are equivalent only if they are used designatively as names of the same man, which is not the case with the sentence "'Cicero' contains six letters" where it is the word *Cicero* itself, not its reference, that is the subject of the sentence. This attempted solution is disputed by Benson Mates in his essay on synonymy, since it does not appear to explain how the sentences "Jones said he has one nose", "Jones said he has  $-e^{\pi i}$  noses" may obviously be not both true even though  $-e^{\pi i} = 1$ .

The tenth essay, by Rudolf Carnap, is outstanding for the maturity and clarity of its thought and the brilliance of its technique. Whether or not one likes or accepts Carnap's views, this essay does arrive at definite con-

clusions, which is more than can be said of most of the contributions. Another essay which is significant for the felicity of its language is Nelson Goodman's "On likeness of meaning", though the conclusion there reached that *no two different words have the same meaning* seems to need further consideration, since there seems to be no obstacle to our inventing a word and declaring it to have a given meaning. The fact that the sentences "*A believes grass is green*" and "*A believes grass is neerg*", where "*neerg*" is the invented word for "*green*", may have different truth values does not oblige us to say that "*green*" and "*neerg*" have different meanings since "*A believes that grass is green*" and "*A believes that grass is not green*" may both be true, if the criterion for their truth is just that *A* assents to both the propositions. If the content of a man's mind is a fit subject matter for logic then both the logical and the illogical thinkers, not just the educated and the ignorant, have a claim to our attention. In fact, one of the essays, "Towards a theory of interpretation and preciseness", by Arne Naess, presents the view that to find out whether two words have the same meaning or not you must run round asking everyone what the words mean to them. However interesting a sociological experiment this may be it surely has no more claim to be considered logical analysis than a schoolboy's faulty arithmetic may be held to test the validity of the multiplication table.

In the last essay Morton V. White seeks to show that the now fashionable dualism of analytic and synthetic sentences is invalid; not of course that  $2+2=4$  is synthetic but that "All men are rational animals" may be.

No collection of essays on Semantics could with propriety fail to contain some contribution by the great Polish logician Alfred Tarski; and in fact Linsky gives Tarski's "The semantic conception of truth" pride of place. Unfortunately the essay chosen is one in which Tarski discusses criticisms of his definition of truth, but the definition in question is given only in quite inadequate outline. No one who is not familiar with the definition—and Tarski himself says that he would not attempt to give it in full without the aid of the whole apparatus of modern symbolic logic—will find much point either in the criticisms which Tarski quotes or in the arguments with which he rebuts them.

R. L. GOODSTEIN.

**Die Praktische Behandlung von Integralgleichungen.** By H. BÜCKNER. (Ergebnisse der angewandten Mathematik, Heft 1). Pp. vi, 127. DM. 18.60. 1952. (Springer-Verlag, Berlin-Göttingen-Heidelberg)

The theory of integral equations had its first great triumphs in the early years of the present century, when it was found to be a tremendously powerful tool for proving general results about the solutions of the linear partial differential equations of mathematical physics: for instance, existence and uniqueness theorems and theorems on the expansion of an arbitrary function in terms of the characteristic functions of a differential system. Many enthusiasts hoped that integral equations would displace differential equations in the practical as well as the theoretical field, but these hopes were disappointed, for the formulae for the solution of a general integral equation turned out, in most cases, to lead to impossibly heavy computations.

In recent years there has been a revival of interest in integral equations as a practical method of solving problems; for this there are two main reasons. One is the development of electronic computing machines, which may become sufficiently powerful to deal directly with the solution of integral equations, and the other is the gradual discovery by a number of investigators of new numerical methods of handling integral equations.

It is to the second of these topics that the present report is devoted. Dr.

Bückner has confined his attention to Fredholm equations of the second kind, i.e., equations of the form

$$\phi(x) = f(x) + \lambda \int_a^b K(x, y) \phi(y) dy,$$

where  $K(x, y)$  (the "kernel") and  $f(x)$  are given functions,  $\lambda$  is a given number, and  $\phi(x)$  is the unknown function to be determined. As the author remarks, many of the methods described in this report are also applicable, with or without modification, to integral equations of other types.

A summary of the Fredholm theory occupies the first chapter, while Chapter II is devoted to methods for the calculation of characteristic values, especially to those depending on extremal properties of various kinds. The recent work of Wielandt in this field is fully taken into account. In the next two chapters we find the two main groups of methods for dealing with integral equations numerically, those depending on iterative processes, and those in which the kernel is replaced by an approximating kernel that has in some respect a simpler structure. The final chapter deals with certain types of kernel for which special methods are available.

Under both the main headings a great variety of methods are described. The method to be chosen for a particular equation depends firstly on the equation itself; as one proceeds from general kernels to normal, Hermitian and definite kernels in turn, more and more methods become applicable. In the second place, the choice of method depends on how much information one wants; whether, for instance, one wants the first characteristic value only or the first few, or the characteristic functions as well, or the solution of the non-homogeneous equation. The well chosen numerical examples will enable the reader to judge to some extent the relative merits of the methods described; it would have been helpful if the author himself had provided a summary giving a comparative evaluation of the various methods for obtaining various amounts of information. Nevertheless, this report will be of great value to anyone who is concerned with the problem of extracting numerical information from integral equations.

This book is the first of a new Springer series, the "Ergebnisse der angewandten Mathematik", and we look forward to a succession of interesting and valuable reports on various topics in applied mathematics.

F. SMITHIES.

**Inequalities.** By G. H. HARDY, J. E. LITTLEWOOD and G. PÓLYA. 2nd edition. Pp. xii, 324. 27s. 6d. 1952. (Cambridge University Press)

A second edition of this well-known work, which first appeared in 1934, has been brought out by the surviving authors, Littlewood and Pólya. It has evidently stood well the test of time, for few changes have been made. Three appendices, containing some recent work on the subject, have been added. One of them illustrates the theory of maxima and minima of functions of several variables by using this method to give still another proof of "Hilbert's inequality". This and "Kronecker's theorem" must be the two most proved theorems in mathematics.

E. C. T.

**Basic Mathematics of Technology. II.** By J. CHANCE and G. F. SIMS. Pp. ix, 282. 12s. 6d. 1952. (Cambridge University Press)

Volume II completes the course, which is intended to satisfy the requirements of the  $S_2$  stage of the National Certificate. The authors have a sound appreciation of underlying principles and they have developed the course with skill and thoroughness. They believe that accurate application of principles can be acquired only by working through a fairly large number of examples and most teachers would agree with this. The book can be recommended for pupils in Secondary Technical Schools and others who are preparing for the

$S_2$  stage. Although the authors claim that the book would be useful for Grammar School pupils who are preparing for the "Alternative Syllabus", it should be pointed out that the book has not sufficient breadth for this purpose, and there are obvious gaps, particularly in geometry, which prevent the book serving as a complete textbook for Grammar Schools. The explanatory matter throughout is cut down to a minimum. S. I.

**Elements of statistics.** By C. G. LAMBE. Pp. viii, 112. 8s. 6d. 1952 (Longmans Green)

Dr. Lambe is on the staff of the Military College of Science, Shrivenham, and has written this book primarily for the use of engineering students. It is a workmanlike production, amply furnished with practical examples.

The first eight chapters are largely on traditional lines, dealing with distributions, both single-valued and bivariate, with a chapter on probability and one on analysis of variance. None of these goes very deeply into the subject: the examples are often of the type of London Intermediate or London B.Sc. Eng. Part I (there is one example from London B.Sc. Eng. Part II), and the bookwork is straightforward—the Gamma distribution and the Beta distribution are placed, with the only reference to moment generating functions, among the examples of Chapter 3.

Chapter 9 deals more fully than is usual in elementary texts on statistics with the theory of errors. This outlines the method of least squares as appropriate to linear functions, and includes a section on undetermined multipliers. Chapter 10 gives an outline of Quality Control, including the control charts for mean, for range, and for fraction defective.

The examples are numerous and, on the whole, of a character presumably more suited for the students with whom Dr. Lambe works than for the school-boy. There is a large number of examples from gunnery, together with an exposition of how the bookie makes his book, and data from poker, roulette and boule. The gunnery tradition comes out in the rather greater emphasis than usual among statisticians that is laid on the probable error and in the reference to the 50% zone, the practical 100% zone, etc. Answers are given. It is not clear how on p. 37 the graph gives the median, and, although good examples on large populations are not too easy to come by, the double-entry table for the correlation of height and of weight is a little unfortunate, for the weight frequencies, either for the height arrays or for the marginal distribution, are not normal, and the regressions are only approximately linear. The proof in Section 8.1 that variances are additive seems unduly heavy.

The book is well printed and bound, and we have noticed no misprint. There are references given for further reading both in the chapters and at the end of the book. Some of the editions referred to in these have been superseded by later ones. The book includes a table of areas of the normal curve, and a four-figure table of squares of numbers.

FRANK SANDON.

**Elementary Principles of Statistics.** By A. C. ROSANDER. Pp. x, 693. 45s. 1951. (D. Van Nostrand, New York; Macmillan, London)

The author of this book is now Statistician to the (U.S.A.) Bureau of Internal Revenue, and was formerly a university lecturer in statistics. He has written this book as "an introduction to the science of statistics", and not as "a book on mathematical statistics". It is a comprehensive treatise, though not one through which I found it easy to make my way. He has set it out in four parts, as follows: Part 1, Basic concepts, pp. 3–24; Part 2, Distribution of measurements, pp. 27–225; Part 3, Distribution of estimates, pp. 229–425; Part 4, Distribution of test statistics, pp. 429–664. The effect of this has been to modify considerably the more usual stages of approach to the topics, and I am not sure how far this modification would be justified on pedagogical

grounds. It has led, for example, to quite a lot about sampling (about, say, stratified sampling, quota sampling, desirable sizes of samples) in Part 2, before correlation, and to approaching confidence limits and the power of a test before  $\chi^2$ , and the ideas of level of significance before rank correlation.

There are 36 chapters in the book. Each consists of text, sometimes with tables of data, of computations (some serially numbered, some not, but usually very fully and carefully worked out), or of specimen forms for lay-out of procedures, with, at the end of each, Questions and Problems, then Experiments, and then References. The text of any chapter often begins extremely simply and gradually, but it is unlikely that the unaided student would get through all the later sections of the chapter. In the text, among the questions and problems, are very extensive series of results and data from all sorts of experiments, observations and experiences from many different fields. This is one of the most valuable parts of the book. The tables include the traditional ones of W. S. Jevons on errors of observation in astronomical work, and of Bortkiewicz on deaths from horse kicks, and, as well, such different matters as records of different determinations of the velocity of light, experiments in short weighing in different types of shop, of church-going among girls (to give a nice parabola for percentage against age), and of the working of the National Draft Lottery (which, it is of interest to note, gave far from random chances for different serial numbers and was grossly unfair to the holders of certain sets of numbers). Several of the examples are used several times in different connections, and many of the results of the reader's own experiments are needed by him for later chapters. The treatment does not involve anything more than school algebra (Hall and Knight's *Higher Algebra* is given among the references, and the method of approximation as  $n \rightarrow \infty$  for the derivation of the Poisson Distribution is of about that standard of rigour, whilst there is no calculus anywhere in the book), so that there is only a passing reference to, e.g., Fisher's Likelihood Function and to his  $k$ -statistics, and nothing at all about, say, moment generating functions.

The text includes a graphical method for determining partial linear regression, due to the author, and something about the author's Inversion Frequency Distribution (which he is too modest to give in the Index), the number of inversions (in ranked lists) being linearly related to Kendall's  $S$  used in similar problems. In the last chapter, particularly, the author stresses the unsolved problems of the science.

The Questions and Problems cannot always be answered from the text itself (e.g., Ch. 27, qn. 9: "Make a report on the life and work of 'Student'"). The Experiments involve work with cards, records abstracted from daily newspapers, random sampling numbers, or tables to be found elsewhere in the book. The References do not always cover all that would be expected for that particular chapter. Thus Yates on Sampling is not to be found in what would appear to be the appropriate chapter, although he is given in the Select Bibliography at the end of the volume. This latter in many cases brings together references previously given in the various chapters, but in some cases it branches out on entirely new lines (e.g., two references under the head "Gambling"). English and other non-American writers are well represented in it.

At the end of the volume is an appendix of seven tables. The proof reading, which throughout the rest of the book has been very good, is here rather disturbing, for the index  $\bar{x}$  of the heading of Table 1 has gone adrift, and a duplicated decimal point in Table 2 hits one in the eye from the second line of the table. Several of the tables appear to have been reproduced from plates used for other publications. The table for  $F$  (from Snedecor) handles rather more values of  $n_1$  and  $n_2$  than are, perhaps, needed in practice. It gives (source not

stated, though perhaps it can be assumed from p. 150 of the text that it is original) Non-repeating Random Numbers, set out in three tables, the first for the range 001-200, the second for the range 001-500, and the last for the range 0001-1000. Some of the tables in the various chapters, both among those numbered and among those not numbered, are really of the general character that we should expect to find in this appendix. Such are those, in particular, of Tables 43 and 44 (pp. 620, 621—it is a pity that there is no ready means of locating on what page any particular table not in the appendix is to be found) about the author's Inversions, and Tables 45A and 45B (pp. 638, 639) from Neyman and Tokarska.

There is an index (pp. 685-693), though, as suggested above, it is not easy to find all the headings likely to be needed. This is, in fact, another aspect of the disadvantage of this useful book. There is such a lot of it that the author must have found considerable difficulty in organising it. It seems from p. 414 that part, at least, of the text was written before 1948: a chart has been lost from p. 303, and there are two versions of a paragraph on p. 625. These are, however, minor points, and once one is accustomed to finding one's way through the book, one will discover that many of the elementary principles of statistics are well and soundly presented in it.

FRANK SANDON.

**Radio. Volume I.** By JOHN D. TUCKER and DONALD WILKINSON. Pp. ix, 177. 7s. 6d. 1952. (English Universities Press)

First of three volumes designed to cover the syllabus of the City and Guilds of London Institute, and also provide a non-examination candidate with a thorough grasp of the basic principles of radio communication, this book can be considered as competent and fulfilling its purpose. Inevitably the contents "seem to tread on classic ground": aërials and tuning, components and valves, A.F. and R.F. amplifiers, power supplies, oscillators, modulation and detection, receivers and measurements; yet a gallant attempt has been made towards a fresh presentation. A newcomer to the subject will, without doubt, find this a useful guide.

From a parochial viewpoint, it is remarkable how thoroughly the work can be covered without any strenuous demand on mathematics. A little work on triangles in connection with vector diagrams is perhaps the most "advanced" item. Correspondingly, there are only a few numerical examples included, most of which involve substitution in an appropriate formula.

F. W. K.

**Daily Life Mathematics. I.** By P. F. BURNS. Pp. 274. 9s. 3d. 1952 (Ginn)

Book I is the first of four books planned for a four years' course of mathematics for Secondary Modern Schools. In the preface, which is an excellent summary of the author's problems, and his aims and methods of achieving them, Mr. Burns emphasizes three points. There is the wide range of intelligence which creates difficult teaching problems; there is the author's conviction that the approach must be a practical one with recourse to visual aids; finally, there is his decision that the course must be built round topics and projects each of which contains "a core of mathematical ideas arranged as progressive studies". The difficulty which attends such a course is that the dull but necessary drill work is apt to be overlooked. To cope with this situation, there is a series of exercises at the end of the book which deal with basic processes. The author recommends recourse to these whenever weakness in basic processes holds up progress.

The author finds that his topics come under two general groups; financial dealings, and space and time ideas. Among the former are Wages and Salaries, House Purchase, Local and Government Finance, the Post Office, and so on.

The space and time topics include Workshop Practice, Practical Surveying, Earth Measurement and Simple Astronomy. Wherever possible, reality and interest are given by short and simple references to the historical developments of ideas and practice. The general result is that the author has produced a stimulating book which deals with everyday topics in an interesting way and he has succeeded in making the subject appear realistic and purposeful to the pupils. In spite of the fact that the book is built round topics, there is a definite development and progression of elementary ideas. Those who heard the author's interesting talk on Astronomy at one of the annual general meetings of the Mathematical Association will not be surprised that there are astronomical illustrations quite early in the course. Ideas are presented in a simple way in a manner suitable to the type of pupil for whom the book is written. Occasionally, over-simplification is conducive to inaccurate thinking. For instance, on p. 197 we have "the barrel of a cable drum is 9". How many yards of cable would unwind in 12 revolutions"? With a cable 3", 2" or even 1", a bad answer would be obtained. On p. 184 we have "A one acre field contains 4840 sq. yd. Show, by multiplication, that the sides are between 69 and 70 yd". We are not told that the field is a square. There are one or two errors of order. No. 15, p. 187 assumes the radius and tangent property which is first mentioned on p. 194. No. 14 requires the midpoint property of a chord although there is no previous reference to this property.

On one vital matter the reader is left in the dark. What are the author's plans, if any, with regard to algebra? So far, he has not tackled the subject. He may, possibly, introduce the subject in the next volume, but, in that case, there might have been room for some simple ideas in Book I. S. I.

**Mathematics To-day. II.** By E. E. BIGGS and H. E. VIDAL. Pp. 358. 8s. 6d. 1952. (Ginn)

In Part II the authors get more into their stride and get to grips with their subject. The opening chapter deals with types of variation and, quite correctly, the subject is developed from considerations of particular cases. The treatment swings back to proportion from the arithmetical and geometrical points of view. An illustration is given of the "Geometric Square" for surveying which is the kind of practical application one is glad to see in a school mathematical book. The authors give applications of principles by a number of homely illustrations. The calculation of great circle sailing is extended to the case of the distance between two points which have different latitudes and longitudes; this is beyond what is usually attempted. Some 20 pages are devoted to sections of a cone. These examples illustrate the fact that appreciable parts of the book follow the personal idiosyncrasies of the authors; these will not always coincide with those of other teachers. There are some good illustrations of solid geometry. The examples on plans and elevations are sufficient to occupy pupils a couple of terms. These are confined almost exclusively to the mechanical drawing of the plans and elevations of different objects. This kind of work is likely to be sterile. There is no suggestion of trying to obtain a rabattement nor is there any general method of finding the length of a line and its inclination to the H.P.

The book is divided into five chapters: "Are Comparisons Odious"; "Different Shapes and Sizes"; "The Measurement of the Earth"; "More Shapes and Sizes"; "All Shapes and Sizes". The sine formula (and applications) comes under "More Shapes and Sizes", and the cosine formula under "All Shapes and Sizes". I leave the reader to puzzle out the logic of this. I should think that one chapter heading would be sufficient, the first, but without the first and last words.

Revision papers containing 119 examples end the book.

S. I.

eying,  
y and  
ments  
ced a  
y and  
to the  
e is a  
heard  
meet-  
re are  
l in a  
ook is  
nking.  
many  
r even  
e field  
een 69  
or two  
which  
chord

thor's  
bject.  
case,  
S. I.  
. 358.

a their  
rectly,  
treat-  
points  
reying  
mathe-  
ber of  
to the  
longi-  
ted to  
parts  
ill not  
ations  
ent to  
to the  
s kind  
tain a  
ne and

ous";  
More  
(and  
rmula  
logic of  
first,  
S. I.

V

H

D  
in  
a  
F  
A  
of  
J

an  
th